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PHH Harmonic Submersions are Stable (*).

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Sunto. – *Si prova che le applicazioni armoniche di tipo PHH sono (debolmente) stabili.*

Summary. – *We prove that PHH harmonic submersions are (weakly) stable.*

1. – Introduction.

A harmonic map between Riemann manifolds is called *(weakly) stable* if the Hessian of the energy functional is (semi) positive definite, see, for example [Urk93], Chapter 5. In particular, an energy-minimizing map is stable. Lichnerowich has proved in 1970 (see [Li70]) that holomorphic maps between Kähler manifolds are (weakly) stable; away from these particular mappings, we do not dispose of many other examples of harmonic maps which are (weakly) stable.

In the joint paper [AAB00], we have introduced a class of harmonic maps, defined on a Riemann manifold, with value in a Kähler manifold, called *PHH harmonic maps* which have a behaviour somewhat similar to that of holomorphic maps. Holomorphic maps between Kähler manifolds are typical examples of PHH harmonic maps, but examples of different flavour have been found in [AA99].

The aim of this paper is to prove that PHH harmonic submersions are actually (weakly) stable, yet another property which relates maps in this class to holomorphic maps (compare to [BBdBR89]). Throughout the paper, we use the notation of [BW03] and [Urk93] ($d\varphi$ for the linear tangent map, $d\varphi^*$ for the adjoint map, etc).

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2. – Preliminaries

2.1 – *Linear Algebra Definitions*

We recall some basic facts on linear algebra used in the sequel.

Let (V, g) and (W, h) two euclidean vector spaces and $L : V \to W$ a linear map.

DEFINITION 1. – *The adjoint operator of L is the map $L^* : W \to V$ characterized by*

$$g(v, L^*(w)) = h(L(v), w), \text{ for all } \ v \in V, w \in W.$$

DEFINITION 2. – *The L-horizontal component of V is the space defined by*

$$H^L := (\operatorname{Ker} L)^\perp = \operatorname{Im} L^*.$$

We denote by g_{H^L} the restriction of the inner product g to H^L.

REMARK 3. – If W is endowed with a complex structure J such that:

(1) $h(JX, JY) = h(X, Y)$, for any $X, Y \in W$;
(2) $\operatorname{Im} L$ is J-invariant;

then on H^L we can define a linear complex structure: $J_H := L^{-1}JL$ and $L : (H^L, J_H) \to (W, J)$ becomes a complex linear map.

REMARK 4. – For the complex linear map $L : (H^L, J_H, g_H) \to (W, J, h)$ defined above, by a simple computation it can be proved that:

$$g_{H^L}(J_H -, J_H -) = g_{H^L}(-, -) \text{ if and only if } LL^*J = JLL^*.$$

2.2 – *PHWC maps* [Lou97], [BBdBR89].

Let $\varphi : (M^m, g) \to (N^{2n}, J, h)$ be a map defined on a Riemann manifold with value in a Kähler one. For any point $x \in M$, we consider $d\varphi_x^* : T_{\varphi(x)}N \to T_xM$ the adjoint of the tangent map $d\varphi_x : T_xM \to T_{\varphi(x)}N$ and $H_x^\varphi := H^{d\varphi_x}$ the horizontal space of $d\varphi$ at x.

If Im $d\varphi_x$ is J-invariant, then one can define an almost complex structure $J_{H,x}$ on the space H_x^φ by

$$J_{H,x} = d\varphi_x^{-1} \circ J_{\varphi(x)} \circ d\varphi_x,$$

see the previous discussion.

Similarly, if the spaces Im $d\varphi_x$ are J-invariant for all x, then we define the almost complex structure on the horizontal distribution H^φ, by $J_H = d\varphi^{-1} \circ J \circ d\varphi$.

DEFINITION 5. – *Notation as before.*

(i) *The map φ is called PHWC (i.e. pseudo-horizontally weakly conformal) at x if and only if* Im $d\varphi_x$ *is J-invariant and $g|_{H^\varphi_x}$ is $J_{H,x}$-Hermitian.*

(ii) *The map φ is called PHWC (pseudo-horizontally weakly conformal) if and only if it is PHWC at any point of X.*

From Remark 4 it follows that the PHWC condition at a point x is equivalent to: $d\varphi_x \circ d\varphi_x^*$ commutes with $J_{\varphi(x)}$.

This notion appears for the first time in [BBdBR89] in relation with stability of minimal immersions.

2.3 – *PHH maps* [AAB00], [AA99].

Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a Kähler manifold, and φ a smooth map from M to N. Denote ∇^M the Levi-Civita connection on M, ∇^N the Levi-Civita connections on N, and $\widetilde{\nabla}$ the induced connection in the bundle $\varphi^{-1}TN$.

DEFINITION 6. – *A map $\varphi : (M^m, g) \to (N^{2n}, J, h)$ from a Riemannian manifold to a Kähler manifold is called PHH (i.e. pseudo-horizontally homothetic) if and only if*

(1) *φ is PHWC;*

(2) *J_H is parallel in horizontal directions (i.e. $\nabla^M_X J_H = 0$ for every $X \in H^\varphi$).*

The above Definition 6 has a local version. If φ is PHWC at x, we say that φ is *PHH at x* if and only if

$$d\varphi_x\big((\nabla^M_v d\varphi_x^*(JY))_x\big) = J_{\varphi(x)} d\varphi_x\big((\nabla^M_v d\varphi_x^*(Y))_x\big)$$

for any horizontal tangent vector $v \in T_xM$, and any vector field Y, locally defined in a neighbourhood of $\varphi(x)$. By definition, a PHWC map is PHH if and only if

$$d\varphi(\nabla^M_X d\varphi^*(JY)) = J d\varphi(\nabla^M_X d\varphi^*(Y)),$$

for any horizontal vector field X on M and any vector field Y on N, i.e. φ is PHH if and only if it is PHH at any point x of M.

This condition emerged as a natural generalization of the horizontal homotheticity. It has a special interest in conjunction with harmonicity, when nice geometric properties are satisfied, [AAB00]. Several non-trivial examples of PHH harmonic submersions can be found in [AA99], [AAB00]. A general recipe for producing harmonic PHH maps is to solve suitable algebraic systems, see [AA99].

3. – The stability of PHH submersions.

In this section we study the stability of a harmonic submersive map $\varphi : (M^m, g) \to (N^{2n}, J, h)$ from a *compact* Riemannian manifold to a Kähler manifold. We know from Theorem 2.1 (a), Proposition 3.1 and Proposition 3.3 in [AAB00] that if φ is PHH, then the fibres of φ are minimal submanifolds. Recall that the stability of harmonic maps is controlled by a condition on the Hessian of the energy-functional, [Urk93], p. 155:

$$H(E)_\varphi(V, V) \geq 0,$$

for any section V of the bundle $\varphi^{-1}TN$.

Let R be the curvature tensor field on N, $\{\varepsilon_1, ..., \varepsilon_m\}$ be an orthogonal vector frame on M, and V be a section in $\varphi^{-1}TN$. Denote by

$$\mathcal{R}_\varphi := \sum_{i=1}^{m} \frac{1}{\|\varepsilon_i\|^2} R(V, d\varphi(\varepsilon_i)) d\varphi(\varepsilon_i)$$

by

$$\overline{\Delta}_\varphi := -\sum_{i=1}^{m} \frac{1}{\|\varepsilon_i\|^2} \left(\widetilde{\nabla}_{\varepsilon_i} \widetilde{\nabla}_{\varepsilon_i} - \widetilde{\nabla}_{\nabla_{\varepsilon_i}\varepsilon_i} \right)$$

the second-order elliptic differential operator called the *rough Laplacian* of φ, (cf. [Urk93], pp. 155), and by

$$\mathcal{J}_\varphi = \overline{\Delta}_\varphi - \mathcal{R}_\varphi.$$

One of the useful properties of the rough Laplacian, which will be constantly used in the sequel is the following, cf. [Urk93], pp. 156.

PROPOSITION 7. – *The rough Laplacian $\overline{\Delta}_\varphi$ satisfies*

$$\int_M h(\overline{\Delta}_\varphi V, W) v_M = \int_M h(\widetilde{\nabla} V, \widetilde{\nabla} W) v_M = \int_M h(V, \overline{\Delta}_\varphi W) v_M,$$

where V and W are sections on $\varphi^{-1}TN$, and

$$h(\widetilde{\nabla} V, \widetilde{\nabla} W) = \sum_{i=1}^{m} \frac{1}{\|\varepsilon_i\|^2} h(\widetilde{\nabla}_{\varepsilon_i} V, \widetilde{\nabla}_{\varepsilon_i} W).$$

We can state and prove now the main result of this paper.

THEOREM 8. – *Let (M^m, g) be a compact Riemann manifold, (N^{2n}, J, h) be a Kähler manifold, and $\varphi : M \to N$ be a harmonic PHH submersion. Then φ is (weakly) stable.*

PROOF. – As in the proof of Theorem 4.1 of [AAB00], we choose a (local) frame

$$\{e_1, ..., e_n, Je_1, ..., Je_n\}$$

in $\varphi^{-1}TN$ such that the system

$$\{d\varphi^*(e_1), ..., d\varphi^*(e_n), d\varphi^*(Je_1), ..., d\varphi^*(Je_n)\}$$

is an orthogonal frame in the horizontal distribution. We also choose $\{u_1, ..., u_s\}$ an orthonormal basis for the vertical distribution. We denote $E_i = d\varphi^*(e_i)$, and $E_i' = d\varphi^*(Je_i)$, for all $i = 1, ..., n$.

With this notation, we apply the same strategy of proof as in [Urk93], pp. 172, Theorem 3.2.

For V a section in $\varphi^{-1}TN$, we apply Proposition 1, and compute:

$$H(E)_\varphi(V, V) = \int_M h(\tilde{\nabla} V, \tilde{\nabla} V) v_M - \int_M h(\mathcal{R}_\varphi V, V) v_M.$$

By definition

$$h(\tilde{\nabla} V, \tilde{\nabla} V) = \sum_{i=1}^{n} \left(\frac{1}{\|E_i\|^2} h(\tilde{\nabla}_{E_i} V, \tilde{\nabla}_{E_i} V) + \frac{1}{\|E_i'\|^2} h(\tilde{\nabla}_{E_i'} V, \tilde{\nabla}_{E_i'} V) \right) + \sum_{j=1}^{s} h(\tilde{\nabla}_{u_j} V, \tilde{\nabla}_{u_j} V).$$

Analogous to the operator used in the proof of Theorem 3.2, Chapter 5, [Urk93], we define, for any $V \in \Gamma(\varphi^{-1}TN)$, the operator $\bar{\partial} V \in \Gamma(\varphi^{-1}TN \otimes \mathcal{H}^*)$, where $\mathcal{H}$ is the horizontal distribution on M, by

$$\bar{\partial} V(X) := \tilde{\nabla}_{J_H X} V - J \tilde{\nabla}_X V,$$

for any X a horizontal vector field on M.

We compute

$$h(\bar{\partial} V, \bar{\partial} V) = \sum_{i=1}^{n} \left\{ \frac{1}{\|E_i\|^2} h(\bar{\partial} V(E_i), \bar{\partial} V(E_i)) + \frac{1}{\|E_i'\|^2} h(\bar{\partial} V(E_i'), \bar{\partial} V(E_i')) \right\}.$$

Since $J_H E_i = E'_i$, $J_H E'_i = -E_i$, and $\|E_i\| = \|E'_i\|$, we obtain

$$h(\bar\partial V, \bar\partial V) = 2\sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \Big(h(\widetilde\nabla_{E_i} V, \widetilde\nabla_{E_i} V) + h(\widetilde\nabla_{E'_i} V, \widetilde\nabla_{E'_i} V) - 2h(\widetilde\nabla_{E'_i} V, J\widetilde\nabla_{E_i} V) \Big)$$

Therefore

$$\int_M \Big(h(\mathcal{J}_\varphi V, V) - \frac{1}{2} h(\bar\partial V, \bar\partial V) \Big) v_M = \sum_{i=1}^{n} \int_M \frac{1}{\|E_i\|^2} \big(2h(\widetilde\nabla_{E'_i} V, J\widetilde\nabla_{E_i} V) - h(R(V, d\varphi(E_i))d\varphi(E_i), V) - h(R(V, d\varphi(E'_i))d\varphi(E'_i), V) \big) v_M.$$

Taking into account the identities $d\varphi(E'_i) = Jd\varphi(E_i)$, $d\varphi(E_i) = -Jd\varphi(E'_i)$, and the basic properties of the curvature tensor field R, we obtain

$$R(V, d\varphi(E_i))d\varphi(E_i) + R(V, d\varphi(E'_i))d\varphi(E'_i) = JR(d\varphi(E_i), d\varphi(E'_i))V,$$

and thus

$$\int_M \Big(h(\mathcal{J}_\varphi V, V) - \frac{1}{2} h(\bar\partial V, \bar\partial V) \Big) v_M$$
$$= \sum_{i=1}^{n} \int_M \frac{1}{\|E_i\|^2} \Big(2h(\widetilde\nabla_{E'_i} V, J\widetilde\nabla_{E_i} V) - h(JR(d\varphi(E_i), d\varphi(E'_i))V, V) \Big) v_M.$$

We compute

$$\begin{aligned} -h(JR(d\varphi(E_i), d\varphi(E'_i))V, V) &= h(R(d\varphi(E_i), d\varphi(E'_i))V, JV) \\ &= h(\widetilde\nabla_{E_i}\widetilde\nabla_{E'_i} V - \widetilde\nabla_{E'_i}\widetilde\nabla_{E_i} V - \widetilde\nabla_{[E_i, E'_i]} V, JV) \\ &= E_i h(\widetilde\nabla_{E'_i} V, JV) - E'_i h(\widetilde\nabla_{E_i} V, JV) \\ &- h(\widetilde\nabla_{\nabla_{E_i} E'_i} V, JV) + h(\widetilde\nabla_{\nabla_{E'_i} E_i} V, JV) \\ &- h(\widetilde\nabla_{E'_i} V, \widetilde\nabla_{E_i} JV) + h(\widetilde\nabla_{E_i} V, \widetilde\nabla_{E'_i} JV). \end{aligned}$$

Similarly to [Urk93], pp. 180, we define a $\mathcal{C}^\infty$ function ϕ on M by the formula:

$$\phi := \sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \big(E_i h(\widetilde\nabla_{E'_i} V, JV) - E'_i h(\widetilde\nabla_{E_i} V, JV) - h(\widetilde\nabla_{\nabla_{E_i} E'_i} V, JV) + h(\widetilde\nabla_{\nabla_{E'_i} E_i} V, JV) \big).$$

Since

$$h(\widetilde\nabla_{E_i} V, \widetilde\nabla_{E'_i} JV) = -h(\widetilde\nabla_{E_i} JV, \widetilde\nabla_{E'_i} V),$$

we have

$$\int_M \Big(h(\mathcal{J}_\varphi V, V) - \frac{1}{2} h(\bar\partial V, \bar\partial V) \Big) v_M = \int_M \phi v_M.$$

The proof of the Theorem will be concluded if we prove

$$\int_M \phi v_M = 0.$$

For this, we use Green's formula. We choose X a horizontal vector field on M defined by the property:

$$g(X, Y) = h(\widetilde{\nabla}_{J_H Y} V, JV),$$

for any vector field Y on M, and we prove $\operatorname{div}(X) = \phi$. Indeed, since the fibres of φ are minimal, and X is horizontal, it follows:

$$\operatorname{div}(X) = \sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \Big(g(E_i, \nabla_{E_i} X) + g(E'_i, \nabla_{E'_i} X) \Big).$$

Next,

$$\begin{aligned}
\operatorname{div}(X) &= \sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \Big(E_i g(E_i, X) - g(\nabla_{E_i} E_i, X) + E'_i g(E'_i, X) - g(\nabla_{E'_i} E'_i, X) \Big) \\
&= \sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \Big(E_i h(\widetilde{\nabla}_{J_H E_i} V, JV) - h(\widetilde{\nabla}_{J_H \nabla_{E_i} E_i} V, JV) \Big) \\
&+ \sum_{i=1}^{n} \frac{1}{\|E_i\|^2} \Big(E'_i h(\widetilde{\nabla}_{J_H E'_i} V, JV) - h(\widetilde{\nabla}_{J_H \nabla_{E'_i} E'_i} V, JV) \Big).
\end{aligned}$$

By the PHH condition, we have

$$J_H \nabla_{E'_i} E'_i = -(\nabla_{E'_i} E_i)^h, \quad \text{and } J_H \nabla_{E_i} E_i = (\nabla_{E_i} E'_i)^h,$$

where by $(\,-\,)^h$ we denoted the horizontal component of $(\,-\,)$, so,

$$\operatorname{div}(X) = \phi.$$

We showed

$$\int_M h(\mathcal{J}_\varphi V, V) v_M = \frac{1}{2} h(\bar\partial V, \bar\partial V) v_M \geq 0,$$

which ends the proof. □

REMARK 9. – Our result improves the main result of [Mo98], provided that the source manifold is compact (condition which is not needed in [Mo98]).

REMARK 10. – As pointed out by the referee, it would be interesting to find sufficient conditions for a stable harmonic submersion from a Riemannian manifold to a Kähler manifold to be pseudo-horizontally homothetic.

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