On a gauge invariant functional

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1 Introduction

From the variational point of view, there are a lot of similarities between the theory of harmonic maps and the theory of Yang-Mills connections.

M. Ara introduced and studied the notion of $f-$harmonicity which is a generalization of harmonicity (see [1] and [2]).

In this paper, we study a gauge invariant functional, $\mathcal{YM}_f$ on the space of all smooth connections D of a vector bundle E over a compact Riemannian manifold (M, g), which is defined by

$$\mathcal{YM}_f(D) = \int\limits_M f\left(\frac{1}{2}\|R^D\|^2\right)\vartheta_g,$$

where $\|R^D\|$ is the norm of the curvature tensor of a connection D and $f : [0, \infty) \to [0, \infty)$ is a function of class C^2 such that $f'(t) > 0$ for any $t \geq 0$. A critical point of $\mathcal{YM}_f$ is called an $f-$Yang-Mills connection. We note that if $f(t) = t$ we obtain the classical Yang-Mills functional (see [3]) and if $f(t) = \exp(t)$ we obtain the exponential Yang-Mills connection (see [5]).

Similar as in [3], we calculate the first and the second variation formulas of the functional $\mathcal{YM}_f$. The main results is the following existence theorem:

Let (M, g) be an n-dimensional compact Riemannian manifold, G a compact Lie group, and E a G-vector bundle over M. Assume that $n \geq 5$ and

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$f''(0) \neq 0$. Then there exists a Riemannian metric $\tilde{g}$ on M which is conformal to g and a G-connection on E such that D is an $f-$ Yang-Mills connection with respect to $\tilde{g}$.

2 Preliminaries

Let P be a principal G-bundle over a compact Riemannian manifold (M, g), where G is a compact Lie group. We denote by E the associated vector bundle to P by a faithful representation $\rho : G \to O(r)$, that is, $E = P \times_\rho \mathbb{R}^r$, where G acts on $P \times \mathbb{R}^r$ by $P \times R^r \ni (u, y) \to (ut, \rho(t)^{-1}y) \in P \times \mathbb{R}^r$, $t \in G$ and its quotient is denoted by $[(u, y)] \in P \times_\rho \mathbb{R}^r$. Each element u of P over $x \in M$ defines a linear isomorphism of $\mathbb{R}^r$ onto the fiber E_x, $u : \mathbb{R}^r \to E_x$ by $y \to [(u, y)]$.

Given a vector bundle F over M we denote by $\Omega^p(F) = \Gamma(\Lambda^p T^*M \otimes F)$ the space of all smooth p-forms on M with values in F, $p \geq 0$.

A connection D on the vector bundle E is defined by specifying a covariant derivative, that is a linear map

$$D : \Omega^0(E) \to \Omega^1(E),$$

such that $D(fs) = df \otimes s + fDs$, for any section $s \in \Omega^0(E)$ and any smooth function $f \in C^\infty(M)$.

Let U an open set of M. For any local trivialisation $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ of E, it can be defined on $\pi^{-1}(U)$ a local frame $\{\tilde{e}_i\}_{i=1}^r$ by $\tilde{e}_i = \psi^{-1}(x, e_i)$ where $x \in M$ and $\{e_i\}_{i=1}^r$ is the standard basis of $\mathbb{R}^r$. Then over U any section s can be written as $s = \sum_{i=1}^r s_i \tilde{e}_i$ with $s_i : U \to \mathbb{R}$, and thus any connection D on E has the local description

$$Ds = \sum_{i,j=1}^{r} (ds_j + A_{ij} s_j) \otimes \tilde{e}_i.$$

The Lie algebra g of G can be identified with a subalgebra of $gl(r, \mathbb{R})$ via the representation ρ. A connection D is called to be a G-connection if the $(r \times r)$-matrix $A = [A_{ij}]$ takes values on g. We denote by $\mathcal{C}(E)$ the space of all smooth G-connections D on E.

Now let $G(E)$ be the gauge group of the vector bundle E, that is the group of all automorphisms of E inducing the identity map of M. The gauge group can be easily identified with the space of smooth sections of the bundle of groups $P \times_{Ad} G$ associated to the adjoint representation Ad of G, which is the group of allautomorphisms φ of P satisfying $\varphi(ua) = \varphi(u)a$, for any $u \in P$ and $a \in G$. We note that there is a natural action of the gauge group $G(E)$ on the space of G-connections $\mathcal{C}(E)$ given by

$$D^{\varphi} = \varphi^{-1} \circ D \circ \varphi, \ D^{\varphi}s := \varphi^{-1}(D(\varphi s)),$$

for any $s \in \Omega^0(E), \varphi \in G(E)$ and $D \in \mathcal{C}(E)$. Related to $G(E)$ is the infinitesimal gauge group or gauge algebra, which can be regarded as the space $\Omega^0(P \times_{Ad} g)$ of smooth sections of the vector bundle $P \times_{Ad} g$, which is identified with a subbundle of the bundle $End(E)$ via the representation ρ, denoted by g_E. The identification is given by:

$$P \times_{Ad} g \ni [(u, A)] \rightarrow u \circ \rho(A) \circ u^{-1} \in End(E).$$

Given a connection D on E, there exists a unique extension of D to $d^D : \Omega^p(E) \longrightarrow \Omega^{p+1}(E)$ such that d^D is linear and for any $\alpha \in \Lambda^q T^* M$ and $\sigma \in \Omega^p(E)$, $d^D(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^q \alpha \wedge d^D \sigma$. For each G-connection D of the vector bundle E, the curvature tensor of D , denoted by R^D, is determined by $(d^D)^2 : \Omega^0(E) \longrightarrow \Omega^2(E)$. It easy to see that $R^D \in \Omega^2(g_E)$. Onthe other hand it holds that

$$R^{D^{\varphi}} = \varphi^{-1} \circ R^D \circ \varphi,$$

for any $\varphi \in \mathcal{C}(E)$.

Let $< , >$ be the inner product on g defined by

$$< A, B >= -\frac{1}{2} tr(\rho(A)\rho(B)) = \frac{1}{2} tr(\rho(A)^t \circ \rho(B)),$$

for any $A, B \in g$, which induces a fibre metric on $P \times_{Ad} g$ and thus a fibre metric on $End(E)$ by

$$< C, D >= \frac{1}{2} tr(C^t \circ D),$$

for any $C, D \in End(E_x), \ x \in M$.

If a vector bundle F over M admits a fiber metric $< , >$, we can define an inner product on $\Lambda^p T_x^* M \otimes F_x$ by

$$< \psi, \varphi >= \sum_{i_1 < \ldots < i_p} < \psi(e_{i_1}, \ldots, e_{i_p}), \varphi(e_{i_1}, \ldots, e_{i_p}) >,$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_xM with respect to the metric g. We denote its norm by $\| \ \|$. Integrating the above pointwise, the inner product over M gives an inner product in $\Omega^p(F)$. Integration on M shall always be with respect to the riemannian volume measure. We then define the operator $\delta^D : \Omega^{p+1}(F) \to \Omega^p(F),\ p \geq 0$, to be the formal adjoint of the operator d^D.

3 The first variation formula

Now let $f : [0, \infty) \to [0, \infty)$ be a function of class C^2 such that $f'(t) > 0$ for any $t \geq 0$. We define the functional $YM_f : \mathcal{C}(E) \to \mathbb{R}$ by

$$YM_f(D) = \int_M f(\frac{1}{2}\|R^D\|^2)\vartheta_g.$$

We note that if $f(t) = t$ the functional above is the clasical Yang-Mills funtional and if $f(t) = \exp(t)$ the functional is the exponential Yang-Mills (see [5]).

It is not dificult to see that

$$\|R^{D^\varphi}\| = \|R^D\|,$$

for any $\varphi \in \mathcal{C}(E)$. Thus the functional YM_f is invariant under the action of the gauge group $G(E)$ on $\mathcal{C}\ (E)$.

In the following we shall calculate the first variation of the functional YM_f.

Theorem 1 *The first variation of the functional YM_f is given by the formula*

$$\frac{d}{dt}|_{t=0} YM_f(D^t) = \int_M < B, \delta^D(f'(\frac{1}{2}\|R^D\|^2)R^D) > \vartheta_g,$$

where

$$B = \frac{d}{dt}|_{t=0} D^t.$$

Consequently, D is a critical point of YM_f if and only if

$$\delta^D(f'(\frac{1}{2}\|R^D\|^2)R^D) = 0.$$

Proof: Let D a G-connection $D \in \mathcal{C}(E)$ and consider a smooth curve $D^t = D + \alpha^t$ on $\mathcal{C}(E)$, $t \in (-\epsilon, \epsilon)$, such that$\alpha^0 = 0$, where $\alpha^t \in \Omega^1(g_E)$. The corresponding curvature is given by

$$R^{D^t} = R^D + d^D\alpha^t + \frac{1}{2}[\alpha^t \wedge \alpha^t],$$

where we define the bracket of g_E-valued 1 forms φ and ψ by the formula $[\varphi \wedge \psi](X,Y) = [\varphi(X), \psi(Y)] - [\varphi(Y), \psi(X)]$ for any vector fields $X, Y \in \Gamma(TM)$. Indeed for any vector fields $X, Y \in \Gamma(TM)$ and $u \in \Gamma(E)$ we have:

$$\begin{aligned}
R^{D^t}(X,Y)(u) &= D^t_X(D^t_Y u) - D^t_Y(D^t_X u) - D^t_{[X,Y]}u = \\
&= D^t_X(D_Y u + \alpha^t(Y)(u)) - D^t_Y(D_X u + \alpha^t(X)(u)) \\
&\quad - D^t_X(D_{[X,Y]}u + \alpha^t([X,Y])(u)) = \\
&= D_X(D_Y u + \alpha^t(Y)(u)) + \alpha^t(X)(D_Y u + \alpha^t(Y)(u)) - \\
&\quad - D_Y(D_X u + \alpha^t(X)(u)) - \alpha^t(Y)(D_X u + \alpha^t(X)(u)) - \\
&\quad - D_{[X,Y]}u - \alpha([X,Y])(u) = \\
&= R^D(X,Y)(u) + D_X(\alpha^t(Y)(u)) - \alpha^t(Y)(D_X u) - \\
&\quad - \Big(D_Y(\alpha^t(X)(u)) - \alpha^t(X)(D_Y u)\Big) - \alpha^t([X,Y])(u) + \\
&\quad + \alpha^t(X)(\alpha^t(Y)(u)) - \alpha^t(Y)(\alpha^t(X)(u)) = \\
&= R^D(X,Y)(u) + (D_X(\alpha^t(Y))(u) - (D_Y(\alpha^t(X))(u) - \\
&\quad - \alpha^t([X,Y])(u) + \frac{1}{2}[\alpha^t \wedge \alpha^t](X,Y)(u) = \\
&= R^D(X,Y)(u) + (d^D\alpha^t)(X,Y)(u) + \frac{1}{2}[\alpha^t \wedge \alpha^t](X,Y)(u)
\end{aligned}$$

Then we have

$$\frac{d}{dt}|_{t=0} f(\frac{1}{2}\|R^{D^t}\|^2) = f'(\frac{1}{2}\|R^D\|^2)\frac{d}{dt}_{|t=0}\frac{1}{2}\|R^{D^t}\|^2 =$$

$$= f'(\frac{1}{2}\|R^D\|^2) < \frac{d}{dt}R^{D^t}, R^D > |_{t=0}$$

$$= f'(\frac{1}{2}\|R^D\|^2) < d^D B, R^D >$$

where $B = \frac{d}{dt}|_{t=0} D^t \in \Omega^1(g_E)$.

Thus we obtain:

$$\frac{d}{dt}|_{t=0} YM_f(D^t) = \int_M f'(\frac{1}{2}\|R^D\|^2) < d^D B, R^D > \vartheta_g =$$

$$= \int_M < B, \delta^D \left(f'(\frac{1}{2}\|R^D\|^2) R^D \right) > \vartheta_g.$$

□

In the case when $f(t) = t$ we remark that a G-connection D isa critical point of the functional YM_f if and only if the curvature tensor R^D is harmonic. Such a connection is called Yang-Mills connection (see[3]).

For the case of the existence of Yang-Mills connection we have the following result of Katagiri (see [4])

Theorem 2 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, P asmooth principal G-bundle over (M, g) with compact group G. Then there exists a conection D_0 and a metric $\tilde{g}$ on M which is conformally equivalent to the original metric g such that D_0 is a Yang-Mills connection with respect to $\tilde{g}$.*

In the following we shall prove an existence theorem for critical points of the functional YM_f.

Theorem 3 *Let (M, g) be an n-dimensional compact Riemannian manifold, G acompact Lie group, and E a G-vector bundle over M. Assume that $n \geq 5$ and $f''(0) \neq 0$. Then there exist a Riemannian metric $\tilde{g}$ on M conformaly equivalent to g and a G-connection D on E such that D is a critical point of the functional YM_f.*

Theorem 3 follows from the theorem 2 and the following theorem:

Theorem 4 *Let (M, g) be an n-dimensional compact Riemannian manifold, G a compact Lie group, and E a G-vector bundle over M. Assume that $n \geq 5$ and $f''(0) \neq 0$ and let D be a Yang-Mills connection. Then there exists a Riemannian metric $\tilde{g}$ on M conformaly equivalent to g such that D is a critical point of the functional YM_f.*

Proof: For a positive C^∞ function σ on M we define $\tilde{g} = \sigma^{-1} g$. If D is an Yang-Mills connection onthe vector bundle E then

$$\delta_g^D R^D = 0 \Leftrightarrow \delta_{\tilde{g}}^D (\sigma^{(n-4)/2} R^D). \tag{1}$$

We suppose that $f''(0) > 0$, the case when$f''(0) < 0$ is similar. Now, as $f''(0) > 0$ and $f \in C^2$, there exist a positive number ϵ such that $f''(t) > 0$ for any $t \in [0, \epsilon)$ and thus f' is invertible on the interval $[0, \epsilon)$ with the smooth inverse $H : [f'(0), f'(\epsilon)) \to [0, \epsilon)$. Thus we have the following relations:

$$H(f'(t)) = t \tag{2}$$

$$H'(f'(t)) f''(t) = 1 \tag{3}$$

for any $t \in [0, \epsilon)$.

We deffine now the smooth function

$$F : [(f'(0))^{2/(n-4)}, (f'(\epsilon))^{2/(n-4)}) \to [0, \epsilon')$$

by

$$F(y) = \frac{H(y^{(n-4)/2})}{y^2}.$$

We shall prove that F is invertible on a certain interval. It is easyto see that

$$F'(y) = \frac{(n-4)H'(y^{(n-4)/2}) y^{(n-4)/2} - 4H(y^{(n-4)/2})}{2y^3},$$

and let $y = (f'(t))^{2/(n-4)}$. Using the relations 2 and3 we get

$$F'((f'(t))^{2/(n-4)}) = \frac{(n-4)f'(t) - 4tf''(t)}{2f''(t)(f'(t))^{6/(n-4)}},$$

for any $t \in [0, \epsilon)$. If we evaluate the above relation on 0, as $n \geq 5$ and $f' > 0$, then there exist a positive number $\epsilon'' \leq \epsilon$ such that $F'((f'(t))^{2/(n-4)}) > 0$

for any $t \in [0, \epsilon'')$, and thus $F : [(f'(0))^{2/(n-4)}, (f'(\epsilon''))^{2/(n-4)}) \to [0, \epsilon''')$ is invertible.

We remark that the metric g can be chosed such that $\|R^D\|_g^2 < \epsilon'''$. Indeed for a positive constant C we define the Riemannian metric g' by $g' = Cg$. Then the Yang-Mills equation for g' is the same as for g. Moreover, since $\|R^D\|_{g'}^2 = C^{-2}\|R^D\|_g^2$ and M is compact, we get $\|R^D\|_g^2 < \epsilon'''$ for C sufficiently large. Now, if we denote by Φ the smooth inverse of F, we define the positive smooth function σ by

$$\sigma = \Phi(\frac{1}{2}\|R^D\|_g^2).$$

Finally from the equation 1 we have

$$0 = \delta_{\tilde{g}}^D(\sigma^{(n-4)/2} R^D) = \delta_{\tilde{g}}^D(\Phi(\frac{1}{2}\|R^D\|_g^2)^{(n-4)/2} R^D) =$$

$$= \delta_{\tilde{g}}^D(f'(\frac{1}{2}\sigma^2\|R^D\|_g^2) R^D) = \delta_{\tilde{g}}^D(f'(\frac{1}{2}\|R^D\|_{\tilde{g}}^2) R^D),$$

which prove that the Yang-Mills connection D is also a critical point of the functional YM_f with respect to the metric $\tilde{g}$. □

4 The second variation formula

In this section we obtain the second variation formula. Let (M, g) be an n-dimensional comapact Riemannian manifold, G a compact Lie group and E a $G-$vector bundle over M. Let D be a critical point of the functional $\mathcal{YM}_f$ and D^t a smooth curve on $\mathcal{C}(E)$ such that $D^t = D + \alpha^t$, where $\alpha^t \in \Omega^t(g_E)$ for all $t \in (-\varepsilon, \varepsilon)$, and $\alpha^0 = 0$. The infinitesimal variation of the connection associated to D^t at $t = 0$ is

$$B := \frac{d\alpha^t}{dt}|_{t=0} \in \Omega(g_E).$$

Define an endomorphism $\mathcal{R}^D$ of $\Omega(g_E)$ following [3] by

$$\mathcal{R}^D(\varphi)(X) := \sum_{i=1}^{n} [R^D(e_i, X), \varphi(e_i)],$$

for $\varphi \in \Omega(g_E)$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on (M, g). Then we obtain:

Theorem 5 *Let (M, g) be an n dimensional compact Riemannian manifold, G a compact Lie group and E a G-vector bundle over M. Let D be an f-Yang-Mills connection on E. Then the second variation of the functional $\mathcal{YM}_f$ is given by:*

$$\frac{d^2}{dt^2}|_{t=0} YM_f(D^t) = \int_M f''(\frac{1}{2}\|R^D\|^2) < d^D B, R^D >^2 \vartheta_g +$$

$$+ \int_M f'(\frac{1}{2}\|R^D\|^2) \Big(< d^D B, d^D B > + < B, \mathcal{R}^D(B) > \Big) \vartheta_g =$$

$$= \int_M < B, \mathcal{S}^D(B) > \vartheta_g,$$

where $\mathcal{S}^D$ is a differential operator acting on $\Omega(g_E)$ defined by:

$$\mathcal{S}^D(B) = \delta^D \Big(f''(\frac{1}{2}\|R^D\|^2) < d^D B, R^D >^2 \Big) + \delta^D \Big(f'(\frac{1}{2}\|R^D\|^2) dB \Big) +$$

$$+ f'(\frac{1}{2}\|R^D\|^2) \mathcal{R}^D(B).$$

The index, nullity and stability of an f-Yang-Mills connection D can be defined in the same way as in the case of Yang-Mills connection.

Corollary 1 *Let D be an f-Yang-Mills connetion of which $\|R^D\|$ is constant and $f'' = f'$. Then the stability as a Yang-Mills connection implies the stability as an f-Yang-Mills connection.*

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