A new class of type Lindley distributions. Model, statistical inference and applications

Phd student: Maria Crina Țuculan (Diaconu)

Scientific coordinator: Prof. Dr. Vasile Preda

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Chapter 1

Introduction

1.1 Motivation

The Lindley distribution was introduced by Lindley (1958) as a new distribution useful to analyze lifetime data but, in spite of little attention in the statistical literature, it is important for studying stress-strength reliability modeling. Ghitany et al. (2008) [71] have discussed various properties of this distribution and showed that in many ways it provides a better model for some applications than the exponential distribution. They also showed in a numerical example that the Lindley distribution gives better modeling for waiting times and survival times data than the exponential distribution.

Besides, some researchers have proposed new classes of distributions based on modifications of the Lindley distribution, including also their properties. In this paper, we propose two new generalizations of the Lindley distribution. We refer to this new generalizations as the exponentiated quasi Lindley distribution (EQL) și and as the beta-exponentiated quasi Lindley distribution (BEQL). They provide more flexibility to analyze complex real data sets. We study some statistical properties for the new distribution.

The work objectives are:
1. Building a new class of type Lindley distributions EQL on the basis of it obtaining the results of this thesis.
2. Establish mathematical properties of this new class
3. The study of stress-strength Reliability of this new class.
4. Make and study the properties of the four parameter BEQL distribution.
5. Some examples of lifetimes data-sets from different fields of knowledge has been considered.
6. Study the goodness-of-fit for this distributions to see the superiority of one over the other.

1.2 Theme actuality

Sankaran [104] introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al. [71] investigated most of the statistical properties of the Lindley distribution, showing this distribution may provide a better fitting than the exponential distribution. Mahmoudi and Zakerzadeh [77] proposed an extended version of the compound Poisson distribution which was obtained by compounding the Poisson distribution with the generalized Lindley distribution which is obtained and analyzed by Zakerzadeh and Dolati [125]. Recently a new extension of the Lindley distribution, called extended Lindley (EL) distribution, which offers a more flexible model for lifetime data is introduced by Bakouch et al. [60]. Adamidis and Loukas [3] introduced a two-parameter lifetime distribution with decreasing failure rate by compounding exponential and geometric distributions, which was named exponential geometric (EG) distribution. In the same way, Kus [68] and Tahmasbi and Rezaei [111] introduced the exponential Poisson (EP) and exponential logarithmic distributions,
respectively. Marshall and Olkin [81] presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Recently, Chahkandi and Ganjali [19] introduced a class of distributions, named exponential powerseries (EPS) distributions, by compounding exponential and power series distributions, where compounding procedure follows the same way that was previously carried out by Adamidis and Loukas [3]; this class contains the distributions mentioned before. Extensions of the EG distribution was given by Adamidis et al. [2] and Barreto-Souza et al. [10], where the last was obtained by compounding Weibull and geometric distributions. A three-parameter extension of the EP distribution was obtained by Barreto-Souza and Cribari-Neto [108]. This new class of distributions has been received considerable attention over the two last years. Weibull power series (WPS), complementary exponential geometric (CEG), two-parameter Poisson-exponential, generalized exponential power series (GEPS), exponentiated Weibull-Poisson (EWP) and generalized inverse Weibull-Poisson (GIWP) distributions were introduced and studied by Morais and Barreto-Souza [84], Louzada-Neto et al. [73], Cancho et al. [18], Mahmoudi and Jafari [78], Mahmoudi and Sepahdar [79] and Mahmoudi and Torki [80].

1.3 Structure of the thesis

The thesis has 6 chapters. The first one is the Introduction.

In the second chapter, we introduce the exponentiated-quasi generalized Lindley distribution (EQL). This chapter is based on the papers [98, 114] and [116]. The chapter has 6 section. In the first section we derive the following properties of the EQL distribution:

1. The relation with the gamma and Weibull distribution
2. Shapes, stochastic orders
3. Moments, conditional moments
4. Entropies: Renyi, Shannon, cumulative residual entropy
5. Quantile function and generation algorithms
6. Extreme values

In the next sections, we discuss the estimation using maximum likelihood method and least square estimation. In the last section we provide two examples of data analysis: model fitting and hypothesis testing.

In Chapter 3 we discuss a model of strength-stress of a system. We derive the reliability of the system in the context of the minimum strength supporting the maximum stress. We calculate the MLE of R and also provide a Monte Carlo algorithm for estimating R. This chapter is based on [48, 98, 114] and [116].

In Chapter 4, we introduce the beta-exponentiated quasi Lindley distribution. This chapter is based on [99, 98, 114] and [116]. It has 9 sections. In the first section we discuss some properties of the BEQL distribution like: expansion of density, some particular cases of the BEQL and hazard and reverse function. In the next section we obtain:

1. Moments
2. Reliability
3. Mean deviations
4. Bonferroni and Lorenz curves
5. Order statistics
6. Entropies: Renyi, Shannon and s-entropy

In the last section we discuss the estimation of the parameters using maximum likelihood method.

In Chapter 5 we introduce a new Lindley type distribution, namely the transmuted exponentiated quasi generalized Lindley distribution. We determine the moments, order statistics, extreme order statistics and discuss the maximum likelihood estimation. This chapter is based on [98, 114, 116, 117] and [100].

The last chapter is about compounding a new generalized Lindley distributions with the discrete distributions: Poisson, binomial and geometric type.
Chapter 2

Exponentiated quasi Lindley distribution

In this chapter we will introduce a new lifetime distribution of type Lindley by considering the exponentiated method. Gupta et al [56] first introduced this method as a generalization of the standard exponential distribution in 1999. We introduce a new three parameter distribution, denoted $EQL(\alpha, \lambda, \beta)$, $\alpha, \lambda, \beta > 0$, referred to as the exponentiated quasi Lindley. This new distribution reduces to the Lindley distribution, the exponential distribution and gamma distribution. On terms of reliability, the various shapes of the $EQL$ distribution give it an advantage, being more suitable to model many real systems which generally exhibit bath-tub shaped failure rate.

**Theorem 2.0.1.** Let $X \sim EQL(\alpha, \lambda, \beta)$. The cumulative function of the $EQL(\alpha, \lambda, \beta)$, $\alpha, \lambda, \beta > 0$, is

$$F(x) = \left[1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x}\right]^\beta$$

(2.0.1)

and the corresponding probability density is given by

$$f(x) = \frac{\beta \lambda^2}{\alpha \lambda + 1} e^{-\lambda x} (\alpha + x) \left[1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x}\right]^{-1}$$

(2.0.2)

**Theorem 2.0.2.** The corresponding hazard rate function is

$$h(x) = \frac{\beta \lambda^2}{\alpha \lambda + 1} e^{-\lambda x} (\alpha + x) \left[1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x}\right]^{-1}$$

(2.0.3)

For $\beta = 1; \beta = 1, \alpha = 1; \beta = 1, \alpha = 0$ and $\beta = 1, \alpha \to \infty$ the model becomes the generalized Lindley model, the Lindley model, the gamma model and the exponential model, respectively.

2.1 Properties

2.1.1 $EQL$ and other distributions. Relations

In this section, we will present some relations between the $EQL$ distribution and the Lindley distribution or the gamma distribution. We have that

**Preposition 2.1.1.**

$$\frac{F_{IT}(x)^\beta}{F_{Gamma}(x)} \leq M \quad \text{and} \quad \frac{f_{IT}(x)^\beta}{f_{Gamma}(x)} \leq M \quad \text{for all } 0 < \eta < \lambda.$$
**Preposition 2.1.2.**
\[
\frac{F_T(x)}{F_{IT}(x)} \leq 1 \text{ for } \alpha \geq 1 \quad \text{and} \quad \frac{F_T(x)}{F_{IT}(x)} \geq 1 \text{ for } \alpha \leq 1.
\]

**Preposition 2.1.3.**
\[
\frac{F(x)}{F_{Gamma}(x)} \leq M \text{ for } \alpha \geq 1, \ 0 < \eta < \lambda.
\]

**Preposition 2.1.4.**
\[
\frac{F_{EGamma}^{\beta}(x)}{F_T(x)} > 1 \text{ for } \eta = \lambda \quad \text{and} \quad \eta = \frac{\lambda}{1 + \alpha \lambda}.
\]

**Preposition 2.1.5.**
\[
\frac{F(x)}{F_{EGamma2}(x)} > 1 \text{ for } \alpha < 1.
\]

**Preposition 2.1.6.**
\[
\frac{F_{Weibull}^{\beta}(x)}{F_{EWeibull}(x)} \leq M_1 \quad \text{and} \quad \frac{f_{Weibull}^{\beta}(x)}{f_{EWeibull}(x)} \leq M_1 \text{ for } 0 < \theta < \min(1, \beta).
\]

**Preposition 2.1.7.**
\[
\frac{F(x)}{F_{EWeibull}(x)} \leq M_1 \text{ for } \alpha \geq 1, \ 0 < \theta < \min(1, \beta).
\]

### 2.1.2 Shapes

In this section, we discuss the shape characteristics of the pdf 2.0.2 and hazard rate function 2.0.4.

Figura 2.1: The probability density, cumulative function and failure rate function of exponentiated quasi Lindley distribution.
Preposition 2.1.8. 
\( f(x) \) is log-concave and has the increasing likelihood property for \( \beta \geq 1 \).

Preposition 2.1.9. We have the following
\[
\frac{d \log h(x)}{dx} = \frac{d \log f(x)}{dx} + \frac{\beta \lambda (\alpha + x)U(x)^{\beta-1} e^{-\lambda x}}{(1 - U(x)^\beta)(\alpha \lambda + 1)}
\]  
(2.1.1)

and
\[
\frac{d^2 \log h(x)}{dx^2} = \frac{d^2 \log f(x)}{dx^2} + \frac{\beta \lambda^2 U(x)^{\beta-1} e^{-\lambda x}}{\alpha \lambda + 1} \times 
\]
\[
\left\{ \left[ 1 - U(x)^\beta \right] \left[ 1 - \lambda (\alpha + x) \right] - \frac{(\beta - 1) \lambda}{\alpha + 1} (\alpha + x)^2 e^{\lambda x} U(x)^{-1} - \frac{\lambda}{1 + \alpha \lambda} (\alpha + x)^2 e^{\lambda x} U(x)^{\beta-1} \right\}
\]  
(2.1.2)

Preposition 2.1.10. \( h(x) \) has a maximum, a minimum and a point of inflection in \( x = x_0 \) that satisfies \( \frac{d \log f(x)}{dx} = 0 \), depending on whether \( \frac{d^2 \log f(x)}{dx^2} < 0 \), \( \frac{d^2 \log f(x)}{dx^2} > 0 \), or \( \frac{d^2 \log f(x)}{dx^2} = 0 \), respectively.

Preposition 2.1.11. The failure rate function \( h(x) \) is IFR on \((0, \frac{1 - \alpha \lambda}{\lambda})\) for \( \beta > 1 \) and \( \alpha \lambda < 1 \).

2.1.3 Stochastic orders

Let \( X_1 \sim EQL(\alpha_1, \lambda_1, \beta_1) \) and \( X_2 \sim EQL(\alpha_2, \lambda_2, \beta_2) \) be two exponentiated quasi Lindley random variables. Let \( F_1 \) denote the cdf of \( X_1 \) and \( F_2 \) the cdf for \( X_2 \).

Preposition 2.1.12. If \( \lambda_1 = \lambda_2 \) and \( \alpha_1 = \alpha_2 \) then \( X_1 \) is stochastically greater than or equal to \( X_2 \), \( F_1(x) \geq F_2(x) \) for all \( x > 0 \) and for all \( 0 < \beta_1 \leq \beta_2 \).

Preposition 2.1.13. If \( \beta_1 = \beta_2 \) and \( \alpha_1 = \alpha_2 \) then \( X_1 \) is stochastically greater than or equal to \( X_2 \), \( F_1(x) \leq F_2(x) \) for all \( x > 0 \) and for all \( 0 < \beta_1 \leq \beta_2 \).

Preposition 2.1.14. If \( \lambda_1 = \lambda_2 \) and \( \alpha_1 = \alpha_2 \) then \( X_1 \) is stochastically greater with respect to likelihood ratio than \( X_1 \) if and only if \( \beta_2 \geq \beta_1 \).

Preposition 2.1.15. If \( \beta_1 = \beta_2 = 1 \) and \( \alpha_1 = \alpha_2 = \alpha \) then \( X_2 \) is stochastically greater with respect to likelihood ratio than \( X_1 \) if and only if \( \lambda_1 \geq \lambda_2 \).

Preposition 2.1.16. If \( \beta_1 = \beta_2 = \beta < 1 \) and \( \alpha_1 = \alpha_2 \) then \( X_2 \) is stochastically greater with respect to likelihood ratio than \( X_1 \) if only if \( \lambda_1 \leq \lambda_2 \).

Preposition 2.1.17. If \( \lambda_1 = \lambda_2 \) and \( \alpha_1 = \alpha_2 \) then \( X_2 \) is stochastically greater than \( X_1 \) with respect to reverse hazard ratio if and only if \( \beta_2 \geq \beta_1 \).
Preposition 2.1.18. If $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$ then $X_2$ is stochastically greater than $X_1$ with respect to reverse hazard ratio if and only if $\lambda_2 \leq \lambda_1$.

Preposition 2.1.19. If $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$ then $X_2$ is stochastically greater than $X_1$ with respect to reverse hazard ratio if and only if $\alpha_1 < \alpha_2$.

If $\beta_1 = \beta_2 = \beta$ and $\lambda_1 = \lambda_2 = \lambda$ then $X_1$ is stochastically greater than $X_2$ with respect to reverse hazard ratio if and only if $\alpha_1 > \alpha_2$.

2.1.4 Moments

Theorem 2.1.1. Let $X$ be a random variable exponentiated quasi Lindley. Then the moment generating function of $X$, $M_X(s)$ is:

$$M_X(s) = \sum_{k=0}^{\beta-1} \frac{\lambda^{k+2}}{(\alpha \lambda + 1)^{k+1}} \left\{ \alpha \left( \frac{\alpha \lambda + 1}{\lambda} \right)^{k+1} \Psi \left( 1, k+2, \frac{\alpha \lambda + 1}{\lambda} \left[ \lambda (k+1-s) \right] \right) + \left( \frac{\alpha \lambda + 1}{\lambda} \right)^{k+2} \right\}$$

Theorem 2.1.2. The $r$th moment of the exponentiated quasi Lindley is

$$E(X^r) = \sum_{k=0}^{\beta-1} \beta^k \frac{\lambda^{k+2}}{(\alpha \lambda + 1)^{k+1}} \left[ \alpha (r+1)! \left( \frac{\alpha \lambda + 1}{\lambda} \right)^{r+k+1} \Psi(r+1, r+2+k, (k+1)(\alpha \lambda + 1)) + (r+2)! \left( \frac{\alpha \lambda + 1}{\lambda} \right)^{r+k+2} \Psi(r+2, r+k+3, (k+1)(\alpha \lambda + 1)) \right]$$

2.1.5 Conditional moments

Theorem 2.1.3. The $r$th conditional moments of the EQL are

$$E(X^r \mid X > x) = \frac{\alpha \lambda^2}{(1 + \alpha \lambda)[1 - \beta(x)]} K(\alpha, \lambda, r, \alpha, \lambda, x)$$

2.1.6 Entropies

Theorem 2.1.4. The Renyi entropy of the EQL distribution is

$$T_r(\gamma) = \frac{\gamma}{1-\gamma} \log \left( \frac{\beta \lambda^2}{1 + \alpha \lambda} \right) + \frac{1}{1-\gamma} \log \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{\gamma \beta - \gamma}{k} \binom{k}{j} \times \left( (-1)^k \frac{\lambda^j}{\lambda} e^{(k+\gamma)\alpha \lambda} \Gamma((\gamma + j + 1, (k+\gamma)\alpha \lambda)(1 + \alpha \lambda)^{(k+\gamma)\lambda} \left[ (k+\gamma)\lambda \right]^\gamma + j+1) \right) \right\}$$
Theorem 2.1.5. The Shannon entropy of the EQL distribution is
\[
H(f) = E[- \log f(X)] = - \log \left( \frac{\beta \lambda^2}{1 + \alpha \lambda} \right) + \lambda E(X) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} E(X^k) + \\
+ (1 - \beta) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{k!}{j!} \frac{(1 + \alpha \lambda)^{k-j} \lambda^j}{(1 + \alpha \lambda)^k} - H(\beta, \lambda, j, \alpha, \lambda + \lambda k)
\] (2.1.6)

Theorem 2.1.6. The cumulative residual entropy of the EQL distribution is
\[
T_c = - \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{j} \left( \beta \right)^j \left( \frac{(-1)^j \lambda^k}{j!(1 + \alpha \lambda)^j} \right) H(\beta i + 1, \lambda, 0, \alpha, \lambda j)
\] (2.1.7)

2.1.7 Quantile function
\[
Q(p) = - \frac{1}{\lambda} \left[ (\alpha \lambda + 1) + W(- A e^{\alpha \lambda + 1}) \right]
\] (2.1.8)

2.1.8 Generation algorithms

Algorithm 1
1. Generate \( U_i \sim \text{Uniform}(0,1), i = 1, n \)
2. Set \( X_i = - \frac{1}{\lambda} \left[ (\alpha \lambda + 1) + W(- A e^{\alpha \lambda + 1}) \right], A = (\alpha \lambda + 1) \left[ 1 - U_i^{1/\beta} \right], i = 1, n \)

Algorithm 2
1. Generate \( U_i \sim \text{Uniform}(0,1), i = 1, n \)
2. Generate \( V_i \sim \text{Exponential}(\lambda), i = 1, n \)
3. Generate \( G_i \sim \text{Gamma}(2, \lambda), i = 1, n \)
4. If \( U_i < p, p = \frac{\alpha \lambda}{\alpha \lambda + 1} \), then set \( X_i = V_i^{1/\beta}, \) otherwise set \( X_i = G_i^{1/\beta}, i = 1, n \).

2.1.9 Extreme values

If we consider a system with \( n \) component and its lifetime \( X_1, X_2, \ldots, X_n \sim \text{EQL}(\alpha, \lambda, \beta), \alpha, \lambda, \beta > 0 \), then \( M_n = \max\{X_1, \ldots, X_n\} \) and \( m_n = \min\{X_1, \ldots, X_n\} \) have the following asymptotic distributions:

Theorem 2.1.7. The asymptotic distribution of \( M_n \) and \( m_n \) are
\[
P(a_n (M_n - b_n) \leq x) \to e^{-e^{-x}}
\] (2.1.9)
and
\[
P(c_n (m_n - d_n) \leq x) \to 1 - e^{-x^{\beta}}, n \to \infty
\] (2.1.10)


2.2 Order statistics

Let $X_1, X_2, ..., X_n$ random variable $EQL(\alpha, \lambda, \beta)$. Let $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$ the corresponding order statistics.

**Theorem 2.2.1.** The density of $X_{k,n}$ is

$$f_{X_{k,n}}(x) = \frac{n!}{(k-1)!(n-k)!} \left[ F(x) \right]^{k-1} \left[ 1 - F(x) \right]^{n-k} f(x) =$$

$$= \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{n-k} \binom{n-k}{i} \left( -1 \right)^i F^{k+i-1}(x)f(x) =$$

$$= \frac{n!}{(k-1)!(n-k)!} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta (k-1)}$$

$$\left[ 1 - \frac{1}{\alpha \lambda + 1} e^{-\lambda x} \right]^{n-k} \cdot \frac{\beta \lambda^2}{\alpha \lambda + 1} e^{-\lambda x} (\alpha + x) \quad (2.2.1)$$

**Theorem 2.2.2.** The corresponding cumulative distribution function of $X_{k,n}$ is

$$F_{X_{k,n}}(x) = \sum_{i=k}^{n} \binom{n}{i} F^i(x) \left[ 1 - F(x) \right]^{n-i} = \sum_{i=k}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \left( -1 \right)^j F^{i+j}(x) =$$

$$= \sum_{i=k}^{n} \sum_{j=0}^{n-i} \left( \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right)^{\beta (i+j)} \quad (2.2.2)$$

**Theorem 2.2.3.** The $r$th moment of $X_{k,n}$ is:

$$E(X_{k,n}^r) = \sum_{j=0}^{n-1} \sum_{i=0}^{k-1} C_n^{i-k} \beta (k+j-1) (-1)^{i+j} \frac{n!}{(k-1)!(n-k)!} \beta \lambda^{i+2} \left[ \frac{\alpha (r+1)!}{\lambda} \right]^{r+1+i} \cdot \Psi(r+1, r+i+2, (i+1)(\alpha \lambda + 1)) + (r+2)! \left( \frac{\alpha \lambda + 1}{\lambda} \right)^{r+i+2} \Psi(r+2, r+i+3, (i+1)(\alpha \lambda + 1)) \quad (2.2.3)$$

2.3 Reliability

We derive the reliability $R$ when $X$ and $Y$ are independent random variables $EQL$ distributed. $X \sim EQL(\alpha_1, \lambda_1, \beta_1)$ and $Y \sim EQL(\alpha_2, \lambda_2, \beta_2)$.

**Theorem 2.3.1.** The reliability $R = P(X > Y)$ is

$$R = \frac{\beta \lambda^2}{1 + \alpha \lambda} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{m} \left( \frac{\beta_1 - 1}{k} \right) \left( \frac{\beta_2}{k} \right) \left( \frac{k}{m} \right) \left( \frac{i}{n} \right) (-1)^{k+i} \frac{\lambda_1^m \lambda_2^n}{(1 + \alpha_1 \lambda_1)^m (1 + \alpha_2 \lambda_2)^n} \times$$

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Let $x_1, x_2, \ldots, x_n$ be a random sample of size $n$ from EQL distribution.

The MLEs $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$ of $\alpha, \lambda, \beta$ are then the solutions of the following non-linear equations:

\[
\frac{\partial \ln L}{\partial \lambda} = \frac{2n}{\lambda} - \frac{n\alpha}{\alpha\lambda + 1} - \sum_{i=1}^{n} x_i + (\beta - 1) \sum_{i=1}^{n} \frac{1}{\alpha\lambda + 1 + \lambda x_i} e^{-\lambda x_i} [\lambda^2 x_i - (\alpha\lambda + 1)(\alpha\lambda + 1 + \lambda x_i) - 1] = 0
\]

\[
\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x_i}{\alpha \lambda + 1} e^{-\lambda x_i} \right] = 0
\]

\[
\frac{\partial \ln L}{\partial \alpha} = -\frac{n\lambda}{\alpha \lambda + 1} + \sum_{i=1}^{n} \left( \frac{1}{\alpha \lambda + 1 + \lambda x_i} - e^{-\lambda x_i} \lambda^2 x_i \right) = 0
\]

(2.4.1)

2.5 Least square estimation

The least square estimation method minimizes the function

\[Q(\alpha, \lambda, \beta) = \sum_{i=1}^{n} (F(x_{i:n}) - \frac{i}{n + 1})^2 .\]

We have the following:

\[
\frac{\partial Q(\alpha, \lambda, \beta)}{\partial \alpha} = 2 \sum_{i=1}^{n} \left[ (1 - \frac{\alpha \lambda + 1 + \lambda x_{i:n}}{\alpha \lambda + 1} e^{-\lambda x_{i:n}})^{\beta - 1} \frac{2 x_{i:n}}{(\alpha \lambda + 1)^2} \right] = 0
\]

\[
\frac{\partial Q(\alpha, \lambda, \beta)}{\partial \beta} = 2 \sum_{i=1}^{n} \left[ (1 - \frac{\alpha \lambda + 1 + \lambda x_{i:n}}{\alpha \lambda + 1} e^{-\lambda x_{i:n}})^{\beta - 1} \ln \left( 1 - \frac{\alpha \lambda + 1 + \lambda x_{i:n}}{\alpha \lambda + 1} e^{-\lambda x_{i:n}} \right) \right] = 0
\]
\[ \frac{\partial Q(\alpha, \lambda, \beta)}{\partial \lambda} = 2 \sum_{i=1}^{n} \left( 1 - \frac{\alpha \lambda + 1 + \lambda x_{i:n} e^{-\lambda x_{i:n}}}{\alpha \lambda + 1} - \frac{i}{n + 1} \right)^{\beta} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x_{i:n} e^{-\lambda x_{i:n}}}{\alpha \lambda + 1} \right]^{\beta - 1} x_{i:n} e^{-\lambda x_{i:n}} \frac{[(\alpha \lambda + 1)(\alpha \lambda + 1 + \lambda x_{i:n}) - 1]}{(\alpha \lambda + 1)^2} = 0 \]

By resolving the above non-linear system we obtain the linear square estimators of \( \alpha, \lambda, \beta, \hat{\alpha}_{LSE}, \hat{\lambda}_{LSE}, \hat{\beta}_{LSE} \).

### 2.6 Data analysis

We fit the EQL distribution to the real data set and compare its fitting with some usual survival distributions. Namely, the Weibull distribution, the Lindley distribution and the generalized Lindley distribution. In this case the exponentiated quasi generalized Lindley distribution is the best fit model. For the next example, hypotheses test will be taken into account.
Chapter 3

Stress-strength Reliability of Exponentialized Quasi Lindley distribution

3.1 Notation and preliminaries

In this chapter, we deal with a new three-parameter lifetime distribution by compounding the Lindley and exponential distributions based on the new class of distribution.

3.2 Reliability

Theorem 3.2.1.

\[
R = n \frac{\beta_2 \theta_2^2}{\alpha_2 \theta_2 + 1} \int_0^\infty \left\{ -\alpha_1 \theta_1 + 1 + \theta_1 y e^{-\theta_1 y} \right\}^{-\beta} e^{-\theta_2 (\alpha_2 + y) A_2 n \beta_2 - 1} dy
\]

3.3 Estimation of the parameters

Theorem 3.3.1.

\[
L(u, v / \lambda) = \left\{ \frac{m \beta_1 \theta_1^2}{\alpha_1 \theta_1 + 1} \right\}^{n_1} e^{-\theta_1} \left\{ \prod_{i=1}^{n_1} (\alpha_1 + u_i) \prod_{i=1}^{n_1} A_i \beta_1 - 1 \prod_{i=1}^{n_1} (1 - A_i \beta_1) \right\}^{m-1} \times \left\{ \frac{n \beta_2 \theta_2^2}{\alpha_2 \theta_2 + 1} \right\}^{n_2} e^{-\theta_2} \left\{ \prod_{i=1}^{n_2} (\alpha_2 + v_i) \prod_{i=1}^{n_2} A_2 \beta_2 - 1 \right\}
\]

Theorem 3.3.2. The reliability R of the re-modeled system using the MLE is

\[
\hat{R} = n \frac{\beta_2 \theta_2^2}{\alpha_2 \theta_2 + 1} \int_0^\infty \left\{ -\alpha_1 \theta_1 + 1 + \theta_1 y e^{-\theta_1 y} \right\}^{-\beta} e^{-\theta_2 (\alpha_2 + y) A_2 n \beta_2 - 1} dy
\]

3.4 Bayes estimation

We can use the following full conditional posterior distributions:

\[
\pi_i(\lambda_i \setminus u, v, \lambda_j, j \neq i) \propto \pi_i(\lambda_i) L(u, v / \lambda)
\]

Using the following Gibbs algorithm, we can estimate R both using bayesian analysis, but also MLE. And, so we can compare the two.
Chapter 4

Beta-exponential quasi Lindley (BEQL) distribution

4.1 Notation and preliminaries

In this chapter we study the properties of the four parameter beta-exponential quasi-Lindley distribution (BEQL). This framework is not only a generalization of the exponential quasi-Lindley distribution but also present preferable properties of increasing, decreasing and bathtub-shaped hazard function.

4.2 The BEQL distribution

Theorem 4.2.1. The five-parameter BEQL cdf is given by
\[
F_{BEQL} (x; \alpha, \lambda, \beta, a, b) = \int_0^{F_{EQL}(x; \alpha, \lambda, \beta)} t^{a-1} (1-t)^{b-1} dt
\]
where \( x > 0, \alpha > 0, \lambda > 0, \beta > 0, a > 0, b > 0 \).

The corresponding BEQL probability density function
\[
f_{BEQL} (x; \alpha, \lambda, \beta, a, b) = \frac{1}{B(a, b)} \left[ F_{EQL} (x) \right]^{a-1} (1 - F_{EQL} (x))^{b-1} f_{EQL} (x) = \]
\[
= \beta \lambda^2 e^{-\lambda x} (\alpha + x) \left[ 1 - \frac{\alpha \lambda + \lambda x}{\alpha + 1} e^{-\lambda x} \right]^{\beta a-1} \times
\]
\[
\times \left\{ 1 - \left[ 1 - \frac{\alpha \lambda + \lambda x}{\alpha + 1} e^{-\lambda x} \right]^{\beta b-1} \right\}
\]

4.2.1 Expansion of density

In this subsection, we study the expansion of the BEQL probability density function. There are two cases when \( b \) is a real number and then when it is an integer. For both cases, we consider also the cases when \( a \) is a real non integer number and then when it is an integer.

4.2.2 Some particular cases of the BEQL distribution

In this subsection, we discuss the case of the BEQL distribution for selected values of the parameters \( \beta, a, b \):

- \( \beta = 1 \) In this case we have the beta-quasi-Lindley (BQL) distribution.
- \( \alpha = 1 \) In this case we have the beta-generalized-Lindley (BGL) distribution.
- \( \beta = 1, \alpha = 1 \) In this case we have the beta-Lindley (BL) distribution.
- \( \beta = a = 1 \) In this case we have the EQL distribution.
\( \beta = a = b = 1 \) In this case we have the generalized Lindley distribution with its submodels: the Lindley distribution, the exponential distribution and the gamma distribution.

### 4.2.3 Hazard and reverse functions

**Proposition 4.2.1.** The hazard rate function of the BEQL distribution is

\[
h_{BEQL}(x; \alpha, \lambda, \beta, a, b) = \frac{f_{EQL}(x)[F_{EQL}(x)]^{a-1}[1 - F_{EQL}(x)]^{b-1}}{B(a,b) - B_{F_{EQL}}(a,b)}
\]

**Proposition 4.2.2.** The reverse function of the BEQL distribution is

\[
\tau_{BEQL}(x; \alpha, \lambda, \beta, a, b) = \frac{f_{EQL}(x)[F_{EQL}(x)]^{a-1}[1 - F_{EQL}(x)]^{b-1}}{B_{F_{EQL}}(a,b)}
\]

\( x > 0, \alpha > 0, \lambda > 0, \beta > 0, a > 0, b > 0. \)

### 4.2.4 Other properties of the BEQL distribution

In this subsection we study the monotonicity properties of the BEQL distribution.

### 4.3 The moments of the BEQL distribution

**Theorem 4.3.1.**

\[
\mu_s' = \int_0^{\infty} x^s f_{BEQL}(x; \alpha, \lambda, \beta, a, b) \, dx
\]

When \( b > 0 \) real non-integer and a integer we have:

\[
\mu_s' = \frac{\beta \lambda^2}{a \lambda + 1} \sum_{i=0}^{\infty} c_i \int_0^{\infty} x^s e^{-\lambda x} (a + x) \left[ 1 - \frac{a \lambda + 1 + \lambda x}{a \lambda + 1} e^{-\lambda x} \right]^{\beta(a+i)-1} \, dx
\]

**Theorem 4.3.2.**

If \( \alpha \) non integer. In this case we have, also, \( \beta(a+i) \) non integer, and

\[
\mu_s' = \frac{\beta}{B(a,b)} \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{i_1+1} \sum_{i_3=0}^{(b-1)i_2+i_1+1} \binom{b-1}{i} \binom{i_1+1}{i_2} \binom{i_2+1}{i_3} \times \frac{(-1)^{i+i_1} \Gamma(s+i_3+1)}{(a \lambda + 1)^{s+i_3-i_2-1}(i_1+1)^{s+i_3+1}}
\]

**Theorem 4.3.3.**

If \( \alpha \) integer. In this case we have, also, \( \beta(a+i) \) integer, and \( \beta(a+i) \) in the last equation stops at \( \beta(a+i) - 1 \).

### 4.4 Reliability

We deduce the reliability function \( R \) for two independent identically distributed random variables \( X \sim BEQL(\alpha_1, \lambda_1, \beta_1, a_1, b_1) \) and \( Y \sim BEQL(\alpha_2, \lambda_2, \beta_2, a_2, b_2) \).
Theorem 4.4.1. The reliability R is

\[
R = \frac{\beta_1}{B(a_1, b_1)B(a_2, b_2) \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{i_1}{i_4} \frac{i_6}{i_7} \left( \frac{\beta_1 a_1 - 1}{i_1} \right) \left( \frac{b_1 - 1}{i_2} \right) \left( \frac{\beta_1 i_3}{i_4} \right) \left( \frac{i_4}{i_5} \right) \left( \frac{b_2 - 1}{j} \right) \left( \frac{\beta_2 (a_2 + j)}{i_6} \right) \left( \frac{i_6}{i_7} \right) \times 
\right]

\left[ 1 + \frac{i_2 + i_5 + i_7 + 1}{\lambda_1 (1 + i_1 + i_4) + \lambda_2 i_6} \right]

(4.4.1)

4.5 Mean deviations

We define the mean deviation from the mean \( \delta_1 \) and from the median as \( \delta_2 \).

Theorem 4.5.1. For a, b non-integer we have:

\[
\delta_1(x) = 2\mu F_{BEQL}(\mu) - 2\mu + \frac{2\beta \lambda^2}{(\alpha + 1)B(a, b)} \sum_{i=0}^{\infty} \sum_{i_1=0}^{\infty} (-1)^i i_1 + i_2 \left( a + i - 1 \right) \left( \frac{i_1}{i_2} \right) \left( b - i \right) \times 
\]

\[ L(\alpha, \lambda, \beta(i_2 + 1), 1, \lambda, \mu) \n\]

\[
\delta_2(x) = -\mu + \frac{2\beta \lambda^2}{(\alpha + 1)B(a, b)} \sum_{i_1=0}^{\infty} (-1)^{i_1 + i_2} \left( a + i - 1 \right) \left( \frac{i_1}{i_2} \right) \left( b - i \right) L(\alpha, \lambda, \beta(i_2 + 1), 1, \lambda, M) 
\]

4.6 Bonferroni and Lorenz curves

Theorem 4.6.1. For b real non-integer and a non-integer we have

\[
B(p) = 1 - \frac{\beta \lambda^2}{p \mu (\alpha + 1)B(a, b)} \sum_{i=0}^{\infty} \sum_{i_1=0}^{\infty} (-1)^i i_1 + i_2 \left( a + i - 1 \right) \left( \frac{i_1}{i_2} \right) \left( b - i \right) L(\alpha, \lambda, \beta(i_2 + 1), 1, \lambda, q) 
\]

(4.6.1)

\[
L(p) = 1 - \frac{\beta \lambda^2}{\mu (\alpha + 1)B(a, b)} \sum_{i_1=0}^{\infty} (-1)^{i_1 + i_2} \left( a + i - 1 \right) \left( \frac{i_1}{i_2} \right) \left( b - i \right) L(\alpha, \lambda, \beta(i_2 + 1), 1, \lambda, q) 
\]

(4.6.2)

4.7 Order statistics

In this section, we will study the distribution of order statistics of the BEQL distribution.

Let \( X_1, X_2, ..., X_n \) be a random sample of size \( n \) of the \( BEQL(\alpha, \lambda, \beta, a, b) \). Suppose that \( X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n} \) are the corresponding order statistics.

Theorem 4.7.1. The density function of the kth order statistics \( X_{k:n} \):

\[
f_k(x) = \frac{n!}{(k-1)!(n-k)!} \sum_{i=0}^{n-k} \frac{n-k}{i} (-1)^i \left( \frac{B_{G EQL(x, \beta, \lambda)}(a, b)}{B(a, b)} \right)^k \left( \frac{\beta \lambda^2 e^{-\lambda x}}{(a\lambda + 1)B(a, b)} \right)^x 
\]
\[
\times \left[ V(x) \right]^{\beta a - 1} \left( 1 - V \beta (x) \right)^{b - 1} = \\
= \frac{\beta \lambda^2 n! e^{-\lambda x} (\alpha + x)}{(\alpha \lambda + 1) \left[ B(a, b) \right]^{1 + i_1} (k - 1)! (n - k)! i_1 = 0 i_2 = 0 \left( n - k \right) \binom{b - 1}{i_2} \left[ B_{G_{EQL}(x, \beta, \lambda)} \right]^{(a, b)} \right]^{k - 1 + i_1} x \\
\times \left[ V(x) \right]^{\beta (a + i_2) - 1}
\] (4.7.1)

**Theorem 4.7.2.** The cumulative function of the kth order statistics \( X_{k,n} \) is:

\[
F_k(x) = n \sum_{i_1 = k \atop i_2 = 0}^{n-i_1} \binom{n}{i_1} \binom{n-i_1}{i_2} (-1)^{i_2} \left[ F(x) \right]^{i_1 + i_2} = \\
= \frac{1}{\left[ B(a, b) \right]^{1 + i_1} i_1 = k \atop i_2 = 0} \sum_{i_1 = k \atop i_2 = 0}^{n-i_1} \binom{n}{i_1} \binom{n-i_1}{i_2} (-1)^{i_2} \left[ B_{G_{EQL}(x, \beta, \lambda)} \right]^{i_1 + i_2}
\] (4.7.2)

### 4.8 Entropies

#### 4.8.1 Renyi entropy

**Theorem 4.8.1.** The Renyi entropy of the BEQL distribution is given by

\[
T_k(\gamma) = \frac{1}{1 - \gamma} \left\{ \gamma \log \left( \frac{\beta \lambda^2}{(1 + \alpha \lambda) B(a, b)} \right) + \\
+ \log \left( \sum_{i_1, i_3, i_4 = 0}^{\infty} \frac{i_1 i_3 i_4}{i_2 = 0} \sum_{i_5 = 0}^{i_4} \left( \frac{\beta \lambda^2 - \gamma}{\beta \lambda^2 - \gamma} \right)^{i_1} \left( \frac{\beta \lambda^2 - \gamma}{\beta \lambda^2 - \gamma} \right)^{i_2} \left( \frac{\beta \lambda^2 - \gamma}{\beta \lambda^2 - \gamma} \right)^{i_3} \left( \frac{\beta \lambda^2 - \gamma}{\beta \lambda^2 - \gamma} \right)^{i_4} \right) + \\
+ \log \left( (\alpha \lambda + 1) -1 \right) \left( \frac{i_1 + i_3 + i_4 + i_5}{\lambda^2} (i_1 + i_3 + i_4 + \gamma) \right)^{1/(\gamma + i_2 + i_5 + 1, \lambda (i_1 + i_3 + i_4 + \gamma))} + \\
\right\}
\] (4.8.1)

#### 4.8.2 Shannon entropy

**Theorem 4.8.2.** The Shannon entropy of the BEQL distribution for \( b \) real non integer and a non integer is:
\[ H(f_{BEQL}) = -\log \left( \frac{\beta^2}{(\alpha \lambda + 1) B(a, b)} \right) + \sum_{i_1=1}^{\infty} (-1)^i_1 E(X^i_1) + \lambda E(X) \]

\[ + (\beta \alpha - 1) E[\log V(x)] + (1 - b) E[\log(1 - V^\beta (x))] \]  

(4.8.2)

4.8.3 s-entropy

**Theorem 4.8.3.** The s-entropy of the BEQL distribution is:

\[ H_s(f_{BEQL}(x)) = \begin{cases} 
M(x), & \text{daca } s \neq 1, s > 0 \\
E[- \log f_{BEQL}(X)] & \text{daca } s = 1 
\end{cases} \]  

(4.8.4)

4.9 Maximum likelihood estimation

The MLE’s \( \hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{a}, \hat{b} \) of \( \alpha, \lambda, \beta, a, b \) are the solutions of the following non-linear equations:

\[
\frac{\partial \log L}{\partial \alpha} = -n \frac{\lambda}{\alpha \lambda + 1} + \sum_{i=1}^{n} \frac{1}{\alpha + x_i} = 0 
\]

\[
\frac{\partial \log L}{\partial \beta} = -\frac{n}{\beta} + a \sum_{i=1}^{n} \log[V(x_i)] = 0 
\]

\[
\frac{\partial \log L}{\partial a} = n[\Psi(a + b) - \Psi(a)] + \beta \sum_{i=1}^{n} \log[V(x_i)] = 0 
\]

\[
\frac{\partial \log L}{\partial b} = n[\Psi(a + b) - \Psi(b)] + \sum_{i=1}^{n} \log[1 - V^\beta(x_i)] = 0 
\]

\[
\frac{\partial \log L}{\partial \lambda} = \frac{2n}{\lambda} - \frac{\alpha n}{\alpha \lambda + 1} - \sum_{i=1}^{n} x_i + (\beta a - 1) \sum_{i=1}^{n} \frac{\partial V(x_i)}{V(x_i)} - \beta(b - 1) \sum_{i=1}^{n} \frac{\partial V(x_i)}{[1 - V^\beta(x_i)]} = 0 
\]
Chapter 5

Transmuted exponentiated quasi Lindley distribution

Theorem 5.0.1. Let $X$ be a random variable transmuted exponentiated quasi generalized Lindley with parameters $\alpha > 0, \lambda > 0, \beta > 0$, and $|\gamma| \leq 1$, $X \sim TEQL(\alpha, \lambda, \beta, \gamma)$. Then its density function is

$$f(x) = \frac{\beta \lambda^2}{\alpha \lambda + 1} e^{-\lambda x} (\alpha + x) \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta-1} \left\{ (1 + \gamma) - 2\gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^\beta \right\}$$

and the corresponding cumulative distribution function is

$$F(x) = (1 + \gamma) \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^\beta - \gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{2\beta}$$

$x > 0$ \hspace{1cm} (5.0.1)

5.1 Moments

Let $X$ be a TEQL random variable, $X \sim TEQL(\alpha, \lambda, \beta, \gamma)$, $\alpha, \lambda, \beta > 0$, $|\gamma| \leq 1$.

Theorem 5.1.1. The $k$th moments of $X$ are:

$$E(X^k) = \frac{\beta \lambda^2}{1 + \alpha \lambda} \left\{ (1 + \gamma) H(\beta, \lambda, k, \alpha, \lambda) - 2\gamma H(2\beta, \lambda, k, \alpha, \lambda) \right\}$$

(5.1.1)

5.2 Order statistics

Let $X_1, X_2, ..., X_n$ be random variables $TEQL(\alpha, \lambda, \beta, \gamma)$, $\alpha, \lambda, \beta > 0$, $|\gamma| \leq 1$. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the corresponding order statistics.

Theorem 5.2.1. The density of $X_{k:n}$ is

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} \left[ (1 + \gamma) \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^\beta - \gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{2\beta} \right]^{k-1} \times$$

$$\times \left\{ 1 - \left[ (1 + \gamma) \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta - \gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{2\beta}} \right] \right\}^{n-k} \times$$

$$\times \beta \lambda^2 e^{-\lambda x} (\alpha + x) \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta-1} \left\{ (1 + \gamma) - 2\gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^\beta \right\}$$

(5.2.1)
Theorem 5.2.2. The corresponding cumulative distribution function of $X_{k:n}$ is

$$F_{X_{k:n}}(x) = \frac{1}{\gamma} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\gamma}$$

5.3 Extreme order statistics

We consider a system with $n$ components and its lifetime $X_1, X_2, \ldots, X_n \sim TEQL(\alpha, \lambda, \beta, \gamma)$, $\alpha, \lambda, \beta > 0$, $|\gamma| \leq 1$. Let $M_n = \max\{X_1, \ldots, X_n\}$ and $m_n = \min\{X_1, \ldots, X_n\}$.

Theorem 2.1.8. The asymptotic distribution of $M_n$ and $m_n$ as $n \to \infty$ are:

$$P(a_n(M_n - b_n) \leq x) \to \exp\{- \exp(-\lambda \beta x)\}$$

and

$$P(c_n(m_n - d_n) \leq x) \to 1 - \exp(-x\beta)$$

5.4 Maximum likelihood estimation

The MLEs $\hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{\gamma}$ of $\alpha, \lambda, \beta, \gamma$ are the solutions of the following nonlinear system of equations:

$$\frac{\partial \ln L}{\partial \beta} = \frac{\sum_{i=1}^{n} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\gamma} \left[ 1 - \frac{1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x}}{1 + \gamma - \gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta}} \right]}{\sum_{i=1}^{n} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta}}$$

$$\frac{\partial \ln L}{\partial \gamma} = \sum_{i=1}^{n} \frac{1 - \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta}}{1 + \gamma - \gamma \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta}}$$

$$\frac{\partial \ln L}{\partial \lambda} = (\beta - 1) \sum_{i=1}^{n} \frac{A(x_i)}{1 - \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\beta}} + \sum_{i=1}^{n} \frac{a(x_i)}{\lambda^2 \frac{\alpha \lambda + 1}{\alpha \lambda + 1} e^{-\lambda x}(\alpha + x)}$$

$A(x) = \frac{1}{\gamma} \left[ 1 - \frac{\alpha \lambda + 1 + \lambda x}{\alpha \lambda + 1} e^{-\lambda x} \right]^{\gamma}$
\[
\frac{\partial \ln L}{\partial \epsilon} = (\beta - 1) \sum_{i=1}^{n} \left( B(x_i) \right) - \sum_{i=1}^{n} \frac{b(x_i)}{e^{-\lambda x (\alpha + x)}} \left( 1 + \frac{\rho \lambda + 1 + \lambda x e^{-\lambda x}}{\rho \lambda + 1} \right)^{\beta - 1} B(x_i)
\]

\[
= \frac{\beta}{\beta - 1} \sum_{i=1}^{n} \left( 1 + \frac{\rho \lambda + 1 + \lambda x e^{-\lambda x}}{\rho \lambda + 1} \right)^{\beta - 1} A(x_i)
\]

\[
= \frac{\beta}{\beta - 1} \sum_{i=1}^{n} \left( 1 + \frac{\rho \lambda + 1 + \lambda x e^{-\lambda x}}{\rho \lambda + 1} \right)^{\beta - 1} B(x_i)
\]
Chapter 6

Compound Generalized Lindley distributions: Poisson, Binomial and Geometric type

6.1 Abstract

Three new extensions of well-known families of distributions are proposed. This extensions are obtained by compounding a new generalized Lindley distribution with the discret distributions: Poisson, binomial and geometric. Some characteristics and properties are discussed.

6.2 Introduction

The compounding of different distributions has led to the extension of many well-known families of distributions obtaining more flexible distributions for modeling lifetime data. In recent years, the study of the Lindley distribution [71, 72] has increased due to the necessity of finding a more suitable distribution for lifetime data analysis. Compounding of the generalized Lindley distribution with the Poisson, binomial and geometric distributions, giving the fact that it is a better distribution than the Lindley distribution and therefore better than the exponential one [50], one should expect that this new derived distributions provide a better fit. Most real systems have a increasing/decreasing or most often a unimodal/bath-tub hazard rate, but not a constant hazard rate. The distributions presented in this paper have different failure rate shapes.

6.3 Compound distribution. Poisson type

6.3.1 Generalized Lindley Poisson Max

Theorem 6.3.1. Let \( X \sim \text{generalized Lindley Poisson Max}(\alpha, \theta, \lambda) \). Then the pdf of \( X \) is

\[
f(x) = \frac{\lambda \theta^2 e^{-\theta x} (x + \alpha)}{(\alpha \theta + 1)(e^{\lambda} - 1)} e^{\left[-\frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x}\right]}
\]

and the corresponding cdf is

\[
F(x) = \frac{\lambda \left[1 - \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x}\right]}{e^{\lambda} - 1}, \quad x > 0, \alpha, \theta, \lambda > 0
\]

Theorem 6.3.2. The \( r \)th moment of the generalized Lindley Poisson Max distribution is

\[
E(X^r) = \frac{\theta^2}{e^{-\lambda}} \sum_{k \geq 1} \frac{(-\lambda)^k}{(k - 1)! \alpha \theta + 1} \left[\theta^{k-1} \left(\frac{\alpha \theta + 1}{\theta}\right)^{r+k} \Gamma(r + 1)\Psi(r + 1, r + k + 1, k(\alpha \theta + 1))\right]
\]
\[ + \theta^{k-1} \left( \frac{\alpha \theta + 1}{\theta} \right)^{r+k+1} \Gamma(r+2) \Psi(r+2, r+k+2, k(\alpha \theta+1)) \]

### 6.3.2 Generalized Lindley Poisson Min

**Theorem 6.3.3.** Let \( X \sim \text{generalized Lindley Poisson Min}(\alpha, \theta, n) \). Then the pdf is

\[ f(x) = \frac{\lambda \theta^2 (\alpha + x)}{\alpha \theta + 1} e^{-\theta x} \frac{1}{e^\lambda - 1} \left[ \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right] \]

and the corresponding cdf is

\[ F(x) = \frac{1}{e^\lambda - 1} \left[ e^\lambda - e^{\frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x}} \right], \quad x > 0, \alpha, \theta, \lambda > 0 \]

**Theorem 6.3.4.** The rth moment is

\[ E(X^r) = \sum_{k \geq 1} \frac{\theta^2}{(\alpha \theta + 1)^k} e^{-\theta x} \frac{\lambda^k}{1 - e^{-\lambda (k-1)}} \left[ \alpha \theta^{-1} \left( \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right)^{r+k} \Gamma(r+1, r+k+1, k(\alpha \theta+1)) \right] \]

\[ + \theta^{k-1} \left( \frac{\alpha \theta + 1}{\theta} \right)^{r+k+1} \Gamma(r+2) \Psi(r+2, r+k+2, k(\alpha \theta+1)) \]

### 6.4 Compound distribution. Binomial type

#### 6.4.1 Generalized Lindley Binomial Max

**Theorem 6.4.1.** Let \( X \sim \text{generalized Lindley Binomial Max}(\alpha, \theta, n, p) \). Then the pdf of \( X \) is

\[ f(x) = \frac{\theta^2 (\alpha + x)}{\alpha \theta + 1} e^{-\theta x} \frac{np}{1-q^n} \left[ 1 - p \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right]^{n-1} \]

and the corresponding cdf is

\[ F(x) = \frac{1}{1-q^n} \left[ 1 - p \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right]^{n-1} - q^n, \quad x > 0, \theta, \alpha, p > 0 \]

**Theorem 6.4.2.** The rth moment is

\[ E(X^r) = \frac{\theta^2 np}{(\alpha \theta + 1)(1-q^n)} H(n-1, \theta, r, \alpha, \theta, p) \]
6.4.2 Generalized Lindley Binomial Min

**Theorem 6.4.3.** Let $X \sim \text{generalized Lindley Binomial Min}(\alpha, \theta, n, p)$. Then the cdf of $X$ is

$$F(x) = 1 - \frac{1}{1 - q^n} \left\{ q + p \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right\} - q^n \right\}$$

(6.4.1)

and the pdf is

$$f(x) = \frac{np}{1 - q^n} \left[ q + p \frac{\theta \alpha + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right]^{n-1} f_{GL}(\alpha, \theta)(x), x > 0, \alpha, \theta, p > 0$$

(6.4.2)

**Theorem 6.4.4.** The rth moment of the generalized Lindley Binomial Min distribution is

$$E(X^r) = \frac{np\theta^2}{(1 - q^n)(\alpha \theta + 1)} \sum_{k=0}^{n-1} p^k q^{n-1-k} C_{n-1}^k \left\{ \alpha \theta \left( \frac{\alpha \theta + 1}{\theta} \right)^r \Gamma(r + 1) \right\}$$

$$\cdot \Psi(r + 1, r + 2 + k, \theta(k + 1) \frac{\alpha \theta + 1}{\theta}) + \theta^k \left( \frac{\alpha \theta + 1}{\theta} \right)^{r + 2 + k} \Gamma(r + 2) \Psi(r + 2, r + 3 + k, \theta(k + 1) \frac{\alpha \theta + 1}{\theta})$$

6.5 Compound distribution. Geometric type

6.5.1 Generalized Lindley Geometric Max

**Theorem 6.5.1.** Let $X \sim \text{generalized Lindley Geometric Max}$. Then the pdf of $X$ is

$$f(x) = \frac{p\theta^2 (\alpha + x)e^{-\theta x}}{\alpha \theta + 1} \left[ 1 - (1 - p) \left( 1 - \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right) \right]^{-2}$$

and the corresponding cdf is

$$F(x) = \frac{p}{1 - p} \left[ \frac{1}{1 - (1 - p) \left( 1 - \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right)^{-1}} \right], x > 0, \alpha, \theta, p > 0$$

**Theorem 6.5.2.** The rth moment of the generalized Lindley Geometric Max distribution is
\[ E(X^r) = \sum_{k \geq 1} \frac{kp\theta^2(1 - p)^k}{(\alpha \theta + 1)^k} \frac{1}{(-\theta)^{k-1}} \frac{1}{\alpha^{k+r+1}} \Gamma(k + r) \Psi(k+r,k+r+2,\alpha k\theta) \]

6.5.2 Generalized Lindley Geometric Min

Theorem 6.5.3. Let \( X \sim \text{generalized Lindley Geometric Min} \). Then the pdf is

\[
f(x) = \frac{p\theta^2 e^{-\theta x}}{\alpha \theta + 1} (\alpha + x) \left[ 1 - (1 - p) \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x} \right]^{-2}
\]

and the corresponding cdf is

\[
F(x) = \frac{1}{1 - p} \left[ 1 - \frac{p}{1 - (1 - p) \frac{\alpha \theta + 1 + \theta x}{\alpha \theta + 1} e^{-\theta x}} \right], \quad x > 0, \alpha, \theta, 0 < p < 1
\]

Theorem 6.5.4. The rth moment of the generalized Lindley Geometric Min distribution is

\[
E(X^r) = \sum_{k \geq 1} \frac{kp(1 - p)^k}{(\alpha \theta + 1)^k} \left\{ \frac{1}{\alpha \theta + 1} \frac{(\alpha \theta + 1)^r}{\theta^{r+k}} \Gamma(r+1) \Psi(r+1,k+r+1,k(\alpha \theta + 1)) \right. \\
+ \left. \left( \frac{\alpha \theta + 1}{\theta} \right)^{r+1+k} \Gamma(r+2) \Psi(r+2,r+2+k,k(\alpha \theta + 1)) \right\}
\]
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