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Strong convergence of two finite families of asymptotically pseudocontractive mappings

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Abstract - In this paper, we propose a modified implicit iteration for two finite families of asymptotically pseudocontractive mappings. Further, convergence theorems for approximation of common fixed points of two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings are proved in Banach spaces. Our results extend and improve a number of results in the literature. We also report some preliminary computational results, which illustrate the efficiency of the algorithm.

Key words and phrases : Implicit iteration process, Lipschitzian asymptotically pseudocontractive mapping, strong convergence, common fixed points.

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1. Introduction and preliminaries

Let K be a nonempty subset of a real Banach space E and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we will denote the single values duality mapping by j.

A mapping $T : K \to K$ is said to be

(1) *uniformly L-Lipschitzian* if for all $x, y \in K$ there exists $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall n \geq 1,$$

(2) *nonexpansive* if for all $x, y \in K$,

$$\|Tx - Ty\| \leq \|x - y\|,$$

(3) *asymptotically nonexpansive* if for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall n \geq 1,$$

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where $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$,

(4) *pseudocontractive* (see [1, 10]) if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2,$$

(5) *strongly pseudocontractive* if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \beta\|x-y\|^2,$$

for some $0 < \beta < 1$.

(6) *asymptotically pseudocontractive* (see [12]) if there exists a sequence $\{k_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2, \quad \forall n \geq 1 \tag{1.1}$$

for all $x, y \in K$. Using the result of Kato in [7], inequality (1.1) can equivalently be written as

$$\|x-y\| \leq \|x - y + r[(k_n I - T^n)x - (k_n I - T^n)y]\| \tag{1.2}$$

for all $r > 0$ and $x, y \in K$.

The class of asymptotically pseudocontractive mapping contains the class of asymptotically nonexpansive mappings.

In 2001, Xu and Ori (see [17]) introduced the following implicit iteration process for a finite family of nonexpansive self mappings in Hilbert space. For arbitrary chosen $x_0 \in K$, the sequence $\{x_n\}$ is constructed by the formula

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}$ be a sequence in $(0, 1)$ and $T_n = T_{n\,(mod\,N)}$.

Chen et al. (see [3]) extended the iteration process (1.3) to a finite family of continuous pseudocontractive self mapping and proved strong and weak convergence theorems.

Osilike and Akuchu [11] modified the iteration (1.3) for finite family of asymptotically pseudocontractive mappings. The iteration process can be expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \quad n \geq 1, \tag{1.4}$$

where $n = (k-1)N + i$, $i = 1, 2, \ldots, N$.

Recently Khan et al. (see [8]), defined an implicit iteration process for two finite families of Lipschitzian pseudocontractive mappings. Let $T_i, S_i : K \to K$ $(i = 1, 2, \ldots, N)$ be two families of pseudocontractive mappings. For $x_0 \in K$, construct a sequence $\{x_n\}$ by

$$x_n = \alpha_n x_{n-1} + \beta_n S_n x_{n-1} + \gamma_n T_n x_n, \quad n \in \mathbb{N}, \tag{1.5}$$

where $T_n = T_{n\,(mod\,N)}$, $S_n = S_{n\,(mod\,N)}$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.

Motivated by the works going in this area, we now propose an implicit iteration process for two finite families of asymptotically pseudocontractive mappings. Let K be a nonempty closed convex subset of E satisfying $K + K \subset K$, and $T_i, S_i : K \to K$ $(i = 1, 2, \dots, N)$ be two finite families of asymptotically pseudocontractive mappings. For arbitrarily chosen $x_0 \in K$, construct a sequence $\{x_n\}$ by

$$x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{k(n)} x_{n-1} + \gamma_n T_{i(n)}^{k(n)} x_n + u_n, \tag{1.6}$$

where $n = (k(n) - 1)N + i(n)$ for each $n \geq 1$, $i(n) = 1, 2, \dots, N$, $k(n) \geq 1$ is a positive integer with $k(n) \to \infty$ as $n \to \infty$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.

Remark 1.1. If we take $S_i = T_i$ for all $i = 1, 2, \dots N$, iteration (1.6) reduces to the following iteration

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_{n-1} + \gamma_n T_{i(n)}^{k(n)} x_n + u_n, \tag{1.7}$$

Iteration (1.7) appears to be new one for approximation of fixed points of finite family of asymptotically pseudocontractive mappings.

Remark 1.2. If we take $S_i = I$, the identity mapping for all $i = 1, 2, \dots N$, iteration (1.6) reduced to the following iteration

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n + u_n, \quad n \geq 1, \tag{1.8}$$

where $n = (k(n) - 1)N + i(n)$, $i(n) = 1, 2, \dots, N$. Iteration (1.8) is similar to (3.1), but having error term.

Therefore the iteration (1.6) is a significant generalization of implicit iteration for two finite families of asymptotically pseudocontractive mappings.

In this paper, we prove necessary and sufficient conditions for strong convergence of iteration (1.6) to common fixed point of two finite families of Lipschitz asymptotically pseudocontractive mapping in arbitrary real Banach space. We also obtain some other strong convergence results. Last but not least, in the Section 3 we provide a concrete example and numerically compute the fixed point. The results presented in the paper extend, improve and generalizes the corresponding results of [2, 3, 5, 6, 8, 9, 11, 14, 15, 17, 18].

2. Main results

The following lemmas will be needed for the proof of our main results:

Lemma 2.1. (see [4]) *Let E be a Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a continuous and strong pseudocontraction. Then T has a unique fixed point.*

Lemma 2.2. (see [12]) *Let E be a uniformly convex Banach space and a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\}$ in $[a, b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions*

$$\limsup_{n\to\infty} \|x_n\| \le d, \quad \limsup_{n\to\infty} \|y_n\| \le d, \quad \textit{and} \quad \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is a constant.

Lemma 2.3. (see [16]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1 + b_n)a_n + c_n \quad \textit{for all } n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *if, in addition, there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ as $n \to 0$.*

In what follows, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, \dots, N\}$, the set of first N natural numbers. $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$.

Let E be a real Banach space and let K be nonempty closed convex subset of E satisfying $K + K \subset K$, and let $\{T_i : i \in J\}$, $\{S_i : i \in J\}$ be two families of uniformly Lipschtizian asymptotically pseudocontractive mappings on K with nonempty common fixed point set $\mathcal{F}$, where $\mathcal{F} = \mathcal{F}_{\mathcal{T}} \bigcap \mathcal{F}_{\mathcal{S}}$, $\mathcal{F}_{\mathcal{T}} = \bigcap_{i=1}^{N} F(T_i)$ and $\mathcal{F}_{\mathcal{S}} = \bigcap_{i=1}^{N} F(S_i)$.

Since T_i, S_i $(i \in J)$ are Lipschitzian, therefore there exists constants $L_i, L_i' > 0$ such that $\|T_i^n x - T_i^n y\| \le L_i \|x - y\|$ and $\|S_i^n x - S_i^n y\| \le L_i' \|x - y\|$ for all $x, y \in K$, $n \in \mathbb{N}$ and $i \in J$.

Also, since T_i, S_i $(i \in J)$ are asymptotically pseudocontractive, therefore there exists sequences $\{k_n^{(i)}\}$ and $\{t_n^{(i)}\}$ such that $\langle T_i^n x - T_i^n y, j(x-y)\rangle \le k_n^{(i)} \|x - y\|^2$ and $\langle S_i^n x - S_i^n y, j(x-y)\rangle \le t_n^{(i)} \|x - y\|^2$ for all $x, y \in K$ and $i \in J$. Set $L = \max_{i\in J}(L_i, L_i')$ and $h_n = \max_{i\in\, J}\left(k_n^{(i)}, t_n^{(i)}\right)$.

Before presenting the main results, we first show that the proposed iteration (1.6), is well defined. Let T, S be two uniformly Lipschitzian asymptotically pseudocontractive mappings. For every $v \in K$, define a mapping $W : K \to K$ by

$$Wx = \alpha v + \beta S^n v + \gamma T^n x + u \quad \text{for all } x \in K,$$

where $\alpha, \beta, \gamma \in (0,1)$ with $\gamma L < 1$.

Then for all $x, y \in K$, $j(x-y) \in J(x-y)$, we have

$$\begin{aligned}\langle Wx - Wy, j(x-y)\rangle &= \langle \gamma T^n x - \gamma T^n y, j(x-y)\rangle \\ &= \langle \gamma(T^n x - T^n y), j(x-y)\rangle \\ &= \gamma\langle (T^n x - T^n y), j(x-y)\rangle \\ &\le \gamma L\|x-y\|^2.\end{aligned}$$

Since $\gamma L \in (0,1)$, W is strongly pseudocontractive, which is also continuous, so by Lemma 2.1, it has a unique fixed point $x^* \in K$, that is, $x^* = \alpha v + \beta S^n v + \gamma T^n x^* + u$.

Thus the implicit iteration (1.6) is defined in K for the two finite families $\{T_i\}$, $\{S_i\}$ of uniformly Lipschitzian asymptotically pseudocontractive self mapping on K, provided $\gamma_n L \in (0,1)$ for all $n \in \mathbb{N}$.

Lemma 2.4. *Let $T_i, S_i : K \to K$ $(i \in J)$ be two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F} \neq \emptyset$ and $\sum_{n=1}^{\infty}(h_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n, \gamma_n \le b < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\beta_n < +\infty$, where a, b are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty}\|u_n\| < \infty$. Assume that $\{x_n\}$ is the sequence generated by (1.6). Then*

(i) $\lim_{n\to\infty}\|x_n - p\|$ *exists for all* $p \in \mathcal{F}$,

(ii) $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *exists, where* $d(x_n, \mathcal{F}) = \inf_{p\in\mathcal{F}}\|x_n - p\|$.

Proof. Let $p \in \mathcal{F}$. It follows from (1.6) that

$$\begin{aligned}\|x_n - p\|^2 &= \big\langle \alpha_n x_{n-1} + \beta_n S_{i(n)}^{k(n)} x_{n-1} + \gamma_n T_{i(n)}^{k(n)} x_n + u_n - p, j(x_n - p)\big\rangle \\ &\le \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle + \beta_n\big\langle S_{i(n)}^{k(n)} x_{n-1} - p, j(x_n - p)\big\rangle \\ &\quad + \gamma_n\big\langle T_{i(n)}^{k(n)} x_n - p, j(x_n - p)\big\rangle + \langle u_n, j(x_n - p)\rangle \\ &\le \alpha_n\|x_{n-1} - p\|\|x_n - p\| + \beta_n L\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + \gamma_n h_{k(n)}\|x_n - p\|^2 + \|u_n\|\|x_n - p\| \\ &= (\alpha_n + \beta_n L)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + \gamma_n h_{k(n)}\|x_n - p\|^2 + \|u_n\|\|x_n - p\|,\end{aligned}$$

after simplification, we have

$$\|x_n - p\| \le \frac{\alpha_n + \beta_n L}{1 - \gamma_n}\|x_{n-1} - p\| + \frac{d_{k(n)}}{1 - \gamma_n}\|x_n - p\| + \frac{\|u_n\|}{1 - \gamma_n},$$

where $d_{k(n)} = h_{k(n)} - 1$ for all $k(n) \ge 1$, by condition $\sum_{n=1}^{\infty}(h_{k(n)} - 1) < \infty$, we have $\sum_{n=1}^{\infty} d_{k(n)} < \infty$. Since $0 < a \le \gamma_n \le b < 1$ for all $n \ge 1$, therefore we have

$$\begin{aligned}\|x_n - p\| &\le \left(1 + \frac{\beta_n(L-1)}{1-\gamma_n}\right)\|x_{n-1} - p\| + \frac{d_{k(n)}}{1-\gamma_n}\|x_n - p\| + \frac{\|u_n\|}{1-\gamma_n}\\ &\le \left(1 + \frac{\beta_n(L-1)}{1-b}\right)\|x_{n-1} - p\| + \frac{d_{k(n)}}{1-b}\|x_n - p\| + \frac{\|u_n\|}{1-b}\\ &= \left(1 + \frac{\beta_n(L-1)}{1-b}\right)\left(\frac{1-b}{1-b-d_{k(n)}}\right)\|x_{n-1} - p\| + \frac{\|u_n\|}{1-b-d_{k(n)}}\\ &= \left(1 + \frac{d_{k(n)} + \beta_n(L-1)}{1-b-d_{k(n)}}\right)\|x_{n-1} - p\| + \frac{\|u_n\|}{1-b-d_{k(n)}}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} d_{k(n)} < \infty$, so $d_{k(n)} \to 0$, as $n \to \infty$, and there exists a positive integer n_0 such that $d_k(n) \le \frac{1-b}{2}$ for all $n \ge n_0$, and we have

$$\|x_n - p\| \le \left(1 + \frac{2d_{k(n)} + 2\beta_n(L-1)}{1-b}\right)\|x_{n-1} - p\| + \frac{2\|u_n\|}{1-b}. \tag{2.1}$$

Taking $a_{n+1} = \|x_n - p\|$, $b_n = \frac{2d_{k(n)}+2\beta_n(L-1)}{1-b}$ and $c_n = \frac{2\|u_n\|}{1-b}$, we can see that

$$\sum_{n=1}^{\infty} b_n < \infty, \quad \sum_{n=1}^{\infty} c_n < \infty.$$

It follows from Lemma 2.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$. Taking the infimum over all $p \in \mathcal{F}$ in (2.1), we have

$$d(x_n, \mathcal{F}) \le \left(1 + \frac{2d_{k(n)} + 2\beta_n(L-1)}{1-b}\right)d(x_{n-1}, \mathcal{F}) + \frac{2\|u_n\|}{1-b}.$$

From Lemma 2.3, we obtain that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. This completes the proof. □

Theorem 2.1. *Let $T_i, S_i : K \to K$ $(i \in J)$ be two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F} \ne \emptyset$ and $\sum_{n=1}^{\infty}(h_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \le \alpha_n, \gamma_n \le b < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Then the sequence $\{x_n\}$ generated by (1.6) converges strongly to a member of $\mathcal{F}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Proof. The necessity of condition is obvious. Thus, we will only prove the sufficiency.

Let $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Then from Lemma 2.4 (ii), we have

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. For any given $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists natural number n_1 such that $d(x_n, \mathcal{F}) < \frac{\varepsilon}{3}$ when $n \geq n_1$. Thus there exists $p^* \in \mathcal{F}$ such that for above ε, there exists a positive integer $n_2 \geq n_1$ such that $\|x_n - p^*\| < \frac{\varepsilon}{2}$ as $n \geq n_2$. Thus, for arbitrary $n, m \geq n_2$, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - p^*\| + \|x_m - p^*\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, therefore $\{x_n\}$ is convergent. Suppose $\lim_{n\to\infty} x_n = q$. Since K is closed, so we get $q \in K$, then $\{x_n\}$ converges strongly to q.

It remains to show that $q \in \mathcal{F}$. Notice that

$$|d(q, \mathcal{F}) - d(x_n, \mathcal{F})| \leq \|q - x_n\|, \quad \forall n \in \mathbb{N},$$

since $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we obtain $q \in \mathcal{F}$ This completes the proof. □

Remark 2.1. Theorem 2.1 extends the corresponding results of Guo and Cho in [5].

The following corollaries follow from Theorem 2.1.

Corollary 2.1. *Let $T_i, S_i : K \to K$ $(i \in J)$ be two finite families of uniformly Lipschitzian pseudocontractive mappings such that $\mathcal{F} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \gamma_n \leq b < 1$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Assume that $\{x_n\}$ is the sequence generated by (1.5). Then $\{x_n\}$ converges strongly to a member of $\mathcal{F}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Remark 2.2. Corollary 2.1 improves Theorem 6 of [8].

Corollary 2.2. *Let $T_i : K \to K$ $(i \in J)$ be a family of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F}_{\mathcal{T}} \neq \emptyset$ and $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \gamma_n \leq b < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Assume that $\{x_n\}$ is the sequence constructed by any one of (1.7) and (1.8). Then $\{x_n\}$ converges strongly to a member of $\mathcal{F}_{\mathcal{T}}$ if and only if*

$$\liminf_{n\to\infty} d(x_n, \mathcal{F}_{\mathcal{T}}) = 0.$$

Remark 2.3. A result similar to Corollary 2.2 for single mapping T can be obtained from Theorem 2.1.

Lemma 2.5. *Let $T_i, S_i : K \to K$, $(i \in J)$ be two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F} \neq \emptyset$ and $\sum_{n=1}^{\infty}(h_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $0 < a \leq \alpha_n, \gamma_n \leq b < 1$, $\alpha_n + \gamma_n \leq c < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b, c are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Assume that $\{x_n\}$ be the sequence generated by (1.6). Then $\lim_{n\to\infty} \|x_n - T_l x_n\| = \lim_{n\to\infty} \|x_n - S_l x_n\| = 0$ for all $l \in J$.*

Proof. From Lemma 2.4 (i), we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. We assume that

$$\lim_{n\to\infty} \|x_n - p\| = d. \tag{2.2}$$

Set $v_{k(n)} = \frac{h_{k(n)}-1}{h_{k(n)}}$, from (1.2), we have

$$\begin{aligned}
&\|x_n - p\| \\
&\leq \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n h_{k(n)}} \left[\left(h_{k(n)} I - T_{i(n)}^{k(n)}\right) x_n - \left(h_{k(n)} I - T_{i(n)}^{k(n)}\right) p \right] \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n h_{k(n)}} \left(h_{k(n)} x_n - T_{i(n)}^{k(n)} x_n - h_{k(n)} p + p \right) \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n h_{k(n)}} \left[h_{k(n)} \left\{ \alpha_n \left(x_{n-1} - T_{i(n)}^{k(n)} x_n \right) \right.\right.\right. \\
&\qquad \left.\left.\left. + \beta_n \left(S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n \right) + u_n \right\} \right] \right\| + \frac{1-\alpha_n}{2\alpha_n} v_{k(n)} \left\| T_{i(n)}^{k(n)} x_n - p \right\| \\
&\leq \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n} \left[\alpha_n \left(x_{n-1} - T_{i(n)}^{k(n)} x_n \right) + \beta_n \left(S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n \right) \right.\right. \\
&\qquad \left.\left. + u_n \right] \right\| + \frac{1-\alpha_n}{2\alpha_n} v_{k(n)} \left\| T_{i(n)}^{k(n)} x_n - p \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2} \left(x_{n-1} - T_{i(n)}^{k(n)} x_n \right) + \frac{(1-\alpha_n)\beta_n}{2\alpha_n} \left(S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n \right) \right. \\
&\qquad \left. + \frac{(1-\alpha_n)}{2\alpha_n} u_n \right\| + \frac{1-\alpha_n}{2\alpha_n} v_{k(n)} \left\| T_{i(n)}^{k(n)} x_n - p \right\| \\
&= \left\| x_n - p + \frac{1}{2}(x_{n-1} - p) - \frac{1}{2}(x_n - p) + \frac{\beta_n}{2\alpha_n} \left(S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n \right) \right. \\
&\qquad \left. + \frac{1}{2\alpha_n} u_n \right\| + \frac{1-\alpha_n}{2\alpha_n} v_{k(n)} \left\| T_{i(n)}^{k(n)} x_n - p \right\|
\end{aligned}$$

$$\leq \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| + \frac{\beta_n}{2\alpha_n} \left\| S_{i(n)}^{k(n)} x_{n-1} - T_{i(n)}^{k(n)} x_n \right\|$$
$$+ \frac{1}{2\alpha_n} \|u_n\| + \frac{1 - \alpha_n}{2\alpha_n} v_{k(n)} \left\| T_{i(n)}^{k(n)} x_n - p \right\|$$
$$\leq \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| + \frac{\beta_n}{2\alpha_n} L[\|x_{n-1} - p\| + \|x_n - p\|]$$
$$+ \frac{1}{2\alpha_n} \|u_n\| + \frac{(1 - \alpha_n)}{2\alpha_n} v_{k(n)} L \|x_n - p\|$$
$$\leq \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| + \frac{\beta_n}{2a} L[\|x_{n-1} - p\| + \|x_n - p\|]$$
$$+ \frac{1}{2a} \|u_n\| + \frac{(1 - \alpha_n)}{2a} v_{k(n)} L \|x_n - p\|.$$

Thus,

$$\liminf_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\|$$
$$+ \liminf_{n\to\infty} \frac{\beta_n}{2a} L[\|x_{n-1} - p\| + \|x_n - p\|]$$
$$+ \liminf_{n\to\infty} \frac{1}{2a} \|u_n\| + \liminf_{n\to\infty} \frac{(1 - \alpha_n) v_{k(n)}}{2a} L \|x_n - p\|.$$

Using $\sum_{n=1}^{\infty} \beta_n < +\infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $0 < a \leq \alpha_n \leq b < 1$, since $v_{k(n)} = \frac{h_{k(n)} - 1}{h_{k(n)}} \in (0, 1)$, so $\lim_{n\to\infty} v_{k(n)} = 0$ and from (2.2), we have

$$\liminf_{n\to\infty} \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| \geq d. \tag{2.3}$$

On the other hand, we have

$$\limsup_{n\to\infty} \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| \leq \limsup_{n\to\infty} \left[\frac{1}{2}\|x_n - p\| + \frac{1}{2}\|x_{n-1} - p\| \right] = d, \tag{2.4}$$

from (2.3) and (2.4), we have

$$\lim_{n\to\infty} \left\| \frac{1}{2}(x_n - p) + \frac{1}{2}(x_{n-1} - p) \right\| = d.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{2.5}$$

Thus for any $i \in J$, we have

$$\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0. \tag{2.6}$$

From (1.6)

$$x_{n-1} = \frac{1}{\alpha_n} x_n - \frac{\beta_n}{\alpha_n} S_{i(n)}^{k(n)} x_{n-1} - \frac{\gamma_n}{\alpha_n} T_{i(n)}^{k(n)} x_n - \frac{1}{\alpha_n} u_n.$$

Then

$$\begin{aligned} x_n - x_{n-1} &= \left(1 - \frac{1}{\alpha_n}\right) x_n + \frac{\beta_n}{\alpha_n} S_{i(n)}^{k(n)} x_{n-1} + \frac{\gamma_n}{\alpha_n} T_{i(n)}^{k(n)} x_n + \frac{1}{\alpha_n} u_n \\ &= \frac{\beta_n}{\alpha_n}\big(S_{i(n)}^{k(n)} x_{n-1} - x_n\big) + \frac{\gamma_n}{\alpha_n}\big(T_{i(n)}^{k(n)} x_n - x_n\big) + \frac{1}{\alpha_n} u_n, \end{aligned} \tag{2.7}$$

so,

$$T_{i(n)}^{k(n)} x_n - x_n = \frac{\alpha_n}{\gamma_n}(x_n - x_{n-1}) - \frac{\beta_n}{\gamma_n}\big(S_{i(n)}^{k(n)} x_{n-1} - x_n\big) - \frac{1}{\gamma_n} u_n.$$

Thus

$$\begin{aligned} \big\|T_{i(n)}^{k(n)} x_n - x_n\big\| &\le \frac{\alpha_n}{\gamma_n}\|x_n - x_{n-1}\| + \frac{\beta_n}{\gamma_n}\big\|S_{i(n)}^{k(n)} x_{n-1} - x_n\big\| + \frac{1}{\gamma_n}\|u_n\| \\ &\le \frac{b}{a}\|x_n - x_{n-1}\| + \frac{\beta_n}{a} L\|x_{n-1} - p\| + \frac{\beta_n}{a}\|x_n - p\| + \frac{1}{a}\|u_n\|. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \big\|T_{i(n)}^{k(n)} x_n - x_n\big\| = 0. \tag{2.8}$$

Now

$$\big\|T_{i(n)}^{k(n)} x_n - x_{n-1}\big\| \le \big\|T_{i(n)}^{k(n)} x_n - x_n\big\| + \|x_n - x_{n-1}\|.$$

So, we have

$$\lim_{n\to\infty} \big\|T_{i(n)}^{k(n)} x_n - x_{n-1}\big\| = 0. \tag{2.9}$$

Since for any positive integer $n > N$, we can write $n = (k(n)-1)N + i(n)$, $i(n) \in J$.

Let $\sigma_n = \big\|T_{i(n)}^{k(n)} x_n - x_{n-1}\big\|$. Then from (2.9), we have $\sigma_n \to 0$. Also,

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\le \big\|x_{n-1} - T_{i(n)}^{k(n)} x_n\big\| + \big\|T_{i(n)}^{k(n)} x_n - T_n x_n\big\| \\ &= \sigma_n + \big\|T_{i(n)}^{k(n)} x_n - T_{i(n)} x_n\big\| \le \sigma_n + L\big\|T_{i(n)}^{k(n)-1} x_n - x_n\big\| \\ &\le \sigma_n + L\big\{\big\|T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N}\big\| \\ &\quad + \big\|T_{n-N}^{k(n)-1} x_{n-N} - x_{(n-N)-1}\big\| + \|x_{(n-N)-1} - x_n\|\big\}. \end{aligned} \tag{2.10}$$

Since for each $n > N$, $n = (n-N)(\bmod N)$ and $n = (k(n)-1)N + i(n)$, $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, i.e., $k(n-N) = k(n) - 1$ and $i(n-N) = i(n)$. Therefore

$$\begin{aligned} \big\|T_{i(n)}^{k(n)-1} x_n - T_{n-N}^{k(n)-1} x_{n-N}\big\| &= \big\|T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N}\big\| \\ &\le L\|x_n - x_{n-N}\|, \end{aligned}$$

and

$$\left\|T_{n-N}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\right\| = \left\|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{(n-N)-1}\right\| = \sigma_{n-N},$$

substituting in (2.10), we get

$$\|x_{n-1} - T_n x_n\| \leq \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} + L\|x_{(n-N)-1} - x_n\|.$$

From (2.6) and (2.9), we have

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0. \tag{2.11}$$

It follows from (2.5) and (2.11), that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| \leq \lim_{n\to\infty} \{\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|\} = 0. \tag{2.12}$$

Consequently, for any $i \in J$, from (2.6) and (2.12), we have

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leq (1+L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0\end{aligned}$$

as $n \to \infty$. This implies that the sequence

$$\bigcup_{i=1}^{N} \{\|x_n - T_{n+i}x_n\|\}_{n=1}^{\infty} \to 0 \quad \text{as } n \to \infty.$$

Since for each $l = 1, 2, \cdots, N$, $\{\|x_n - T_l x_n\|\}$ is a subsequence of $\bigcup_{i=1}^{N}\{\|x_n - T_{n+i}x_n\|\}$, therefore, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in J.$$

From (2.7), we have

$$S_{i(n)}^{k(n)}x_{n-1} - x_n = \frac{\alpha_n}{\beta_n}(x_n - x_{n-1}) - \frac{\gamma_n}{\beta_n}(T_{i(n)}^{k(n)}x_n - x_n) - \frac{1}{\beta_n}u_n.$$

Then using conditions $\alpha_n + \beta_n + \gamma_n = 1$ and $\alpha_n + \gamma_n \leq c < 1$, we have

$$\left\|S_{i(n)}^{k(n)}x_{n-1} - x_n\right\| \leq \frac{b}{1-c}\|x_n - x_{n-1}\| + \frac{b}{1-c}\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| + \frac{1}{1-c}\|u_n\|.$$

From (2.5) and (2.8), we have

$$\lim_{n\to\infty} \left\|S_{i(n)}^{k(n)}x_{n-1} - x_n\right\| = 0.$$

Consequently

$$\lim_{n\to\infty} \left\|S_{i(n)}^{k(n)}x_n - x_n\right\| = 0.$$

Repeating above process, we can show that

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \quad \forall l \in J.$$

This completes the proof. □

A mapping $T : K \to K$ is said to be *semicompact* if any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, has a convergent subsequence.

Theorem 2.2. *Let $T_i, S_i : K \to K$ $(i \in J)$ be two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F} \neq \emptyset$ and one of the mappings in $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ is semicompact. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$, $0 < a \leq \alpha_n, \gamma_n \leq b < 1$, $\alpha_n + \gamma_n \leq c < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b, c are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Then the sequence $\{x_n\}$ generated by (1.6) converges strongly to a member of $\mathcal{F}$.*

Proof. From Lemma 2.4, $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$, therefore $\{x_n\}$ is bounded. Without loss of generality, we may assume that T_s is semicompact for some fixed $s \in \{1, 2, \ldots, N\}$. Then by Lemma 2.5, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = \lim_{n\to\infty} \|x_n - S_l x_n\| = 0, \quad \forall l \in J.$$

Therefore $\lim_{n\to\infty} \|x_n - T_s x_n\| = 0$. So by definition of semicompactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to a $x^* \in K$. By continuity of T_l and S_l, we have $T_l x_{n_j} \to T_l x^*$ and $S_l x_{n_j} \to S_l x^*$ for all $l \in J$. Thus by Lemma 2.5, we have

$$\lim_{j\to\infty} \left\|x_{n_j} - T_l x_{n_j}\right\| = \|x^* - T_l x^*\| = 0 \text{ and } \lim_{j\to\infty} \left\|x_{n_j} - S_l x_{n_j}\right\| = \|x^* - S_l x^*\| = 0$$

for all $l \in J$. This implies that $x^* \in \mathcal{F}$. Thus $x_{n_j} \to x^* \in \mathcal{F}$. But $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Therefore

$$\lim_{n\to\infty} x_n = x^* \in \mathcal{F}.$$

This completes the proof. □

If we take $T_i = S_i$ for all $i \in J$, we get following result from Theorem 2.2.

Theorem 2.3. *Let $T_i : K \to K$ $(i \in J)$ be a finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F}_{\mathcal{T}} \neq \emptyset$ and one of the mappings in $\{T_i\}_{i=1}^N$ is semicompact. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying $\alpha_n+\beta_n+\gamma_n = 1$, $0 < a \leq \alpha_n, \gamma_n \leq b < 1$, $\alpha_n + \gamma_n \leq c < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b, c are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Then the sequence $\{x_n\}$ generated by (1.7) converges strongly to a member of $\mathcal{F}_{\mathcal{T}}$.*

Remark 2.4. Theorem 2.2 and Theorem 2.3 extend and generalize corresponding results of [8].

A mapping $T : K \to K$ is said to be *demicompact* if for every bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\}$ converges, there exist a subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ that converges strongly to some x^* in K.

A mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) (see [13]) on K if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0, \infty)$ such that for all $x \in K$,

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T)) = \inf_{x^* \in F(T)} \|x - x^*\|$.

Senter and Dotson established in [13] a relation between condition (A) and demicompactness that the condition (A) is weaker than demicompactness for a nonexpansive mapping T defined on a bounded set. Every compact operator is demicompact. Since every completely continuous mapping $T : K \to K$ is continuous and demicompact, so it satisfies condition (A).

Therefore in the next result, instead of complete continuity of mappings, we use condition that atleast one member satisfy condition (A).

Theorem 2.4. *Let $T_i, S_i : K \to K$ $(i \in J)$ be two finite families of uniformly Lipschitzian asymptotically pseudocontractive mappings such that $\mathcal{F} \neq \emptyset$ and one of the mappings in $\{T_i\}_{i=1}^N$ and $\{S_i\}_{i=1}^N$ satisfy condition (A). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $(0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \gamma_n \leq b < 1$, $\gamma_n L < 1$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, where a, b are some constants. Let $\{u_n\}$ be a bounded sequence in K with $\sum_{n=1}^{\infty} \|u_n\| < \infty$. Then the sequence $\{x_n\}$ defined by (1.6) converges strongly to a member of $\mathcal{F}$.*

Proof. By Lemma 2.4, we see that $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exist.

Let one of the $T_i's$ or $S_i's$ say T_s $(s \in J)$ satisfies condition (A).

Thus by Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_n - T_s x_n\| = 0.$$

So we have $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$.

By the nature of f and the fact that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0,$$

thus there exists a sequence say $\{x_j^*\}$ in $\mathcal{F}$ and subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\left\|x_{n_j} - x_j^*\right\| \leq 2^{-j}, \quad \text{for } j \geq 1.$$

From (2.1), we have

$$\|x_{n_{j+1}} - x_j^*\| \leq (1 + b_{n_j})\|x_{n_j} - x_j^*\| + \frac{2\|u_n\|}{1-b}$$

for a subsequence $\{b_{n_j}\}$ of $\{b_n\}$.

This implies that

$$\|x_{j+1}^* - x_j^*\| \leq \|x_{j+1}^* - x_{n_{j+1}}\| + \|x_{n_{j+1}} - x_j^*\| \leq 2^{-(j+1)} + 2^{-j}(1 + b_{n_j}) + \frac{2\|u_{n_j}\|}{1-b}.$$

Hence $\{x_j^*\}$ is a Cauchy sequence and so converges to some x^* in K. Since $\mathcal{F}$ is closed, x^* is in $\mathcal{F}$ and since $\lim_{n\to\infty}\|x_n - x^*\|$ exists. $\{x_n\}$ converges to x^*. This completes the proof. □

3. Numerical Experiments

In this section, we present some numerical results for the proposed implicit algorithm. We begin with an example of asymptotically pseudocontractive mapping.

Example 3.1. Let $E = \mathbb{R}$ with the usual norm, where $\mathbb{R}$ denote the set of real numbers. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \mu x$, where $\mu \in (0, 1)$. Then,

$$\begin{aligned}\langle T^n x - T^n y, x - y\rangle &= \mu^n |x - y|^2 \\ &< k_n |x - y|^2\end{aligned}$$

where $\{x_n\}$ is a sequence with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$. Hence T is an asymptotically pseudocontractive mapping.

We now consider an example to illustrate the theoretical result:

Example 3.2. Let $E = \mathbb{R}$ with the usual norm. Let $T_m, S_m : \mathbb{R} \to \mathbb{R}$ be mappings given by

$$T_m x = \frac{1}{m+1}\,x \qquad \text{and} \qquad S_m x = \frac{m}{m+1}\,x\,, \qquad \forall x \in \mathbb{R},\, m \in J.$$

In the light of Example 3.1, we can see that each T_m, S_m is asymptotically pseudocontractive mapping with the fixed point 0, hence $\mathcal{F} = \mathcal{F}_{\mathcal{T}} \bigcap \mathcal{F}_{\mathcal{S}} = \{0\}$.

Let $\{x_n\}$ be the sequence defined by (1.6), then

$$x_n - \gamma_n T_{i(n)}^{k(n)} x_n = \alpha_n x_{n-1} + \beta_n S_{i(n)}^{k(n)} x_{n-1} + u_n\,,$$

by the definition of T_m and S_m, we have

$$x_n - \gamma_n \left(\frac{1}{i(n)+1} \right)^{k(n)} x_n = \alpha_n x_{n-1} + \beta_n \left(\frac{i(n)}{i(n)+5} \right)^{k(n)} x_{n-1} + u_n \,,$$

after simplification, we get

$$x_n = \frac{\left[\alpha_n + \beta_n \left(\frac{i(n)}{i(n)+5} \right)^{k(n)} \right] x_{n-1} + u_n}{1 - \gamma_n \left(\frac{1}{i(n)+1} \right)^{k(n)}} \,, \tag{3.1}$$

where $n = (k(n)-1)\,N + i(n)$, $i(n) \in J$.

Using MATLAB, we compute x_n by (3.1). For this we first set $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{1}{(n+1)^2}$, $\gamma_n = 1 - \alpha_n - \beta_n$ and $u_n = \frac{1}{(n+1)^3}$ for all $n \geq 1$. Set the stop parameter to $\|x_n - x^*\| \leq 10^{-5}$. Computational results for $x_0 = 10$, $N = 15$ are reported in Table 1 and Figure 1.

Table 1: The iteration chart

n	iterate value	$\|x_n - x*\|$	n	iterate value	$\|x_n - x*\|$
0	10	0	24	0.0000649800	0.0000649800
1	1.5666725568	1.5666725568	25	0.0000576526	0.0000576526
2	0.1543746046	0.1543746046	26	0.0000514019	0.0000514019
3	0.0269845519	0.0269845519	27	0.0000460324	0.0000460324
4	0.0107571101	0.0107571101	28	0.0000413915	0.0000413915
5	0.0057905644	0.0057905644	29	0.0000373580	0.0000373580
6	0.0035059595	0.0035059595	30	0.0000338347	0.0000338347
7	0.0022862635	0.0022862635	31	0.0000349957	0.0000349957
8	0.0015740578	0.0015740578	32	0.0000290169	0.0000290169
9	0.0011299878	0.0011299878	33	0.0000259427	0.0000259427
10	0.0008386475	0.0008386475	34	0.0000235951	0.0000235951
11	0.0006395510	0.0006395510	35	0.0000216069	0.0000216069
12	0.0004988601	0.0004988601	36	0.0000198655	0.0000198655
13	0.0003966181	0.0003966181	37	0.0000183186	0.0000183186
14	0.0003205294	0.0003205294	38	0.0000169339	0.0000169339
15	0.0002627336	0.0002627336	39	0.0000156882	0.0000156882
16	0.0002731288	0.0002731288	40	0.0000145632	0.0000145632
17	0.0001946919	0.0001946919	41	0.0000135442	0.0000135442
18	0.0001566944	0.0001566944	42	0.0000126186	0.0000126186
19	0.0001311047	0.0001311047	43	0.0000117757	0.0000117757
20	0.0001117764	0.0001117764	44	0.0000110065	0.0000110065
21	0.0000964487	0.0000964487	45	0.0000103030	0.0000103030
22	0.0000839702	0.0000839702	46	0.0000102984	0.0000102984
23	0.0000736388	0.0000736388	47	0.0000091793	0.0000091793

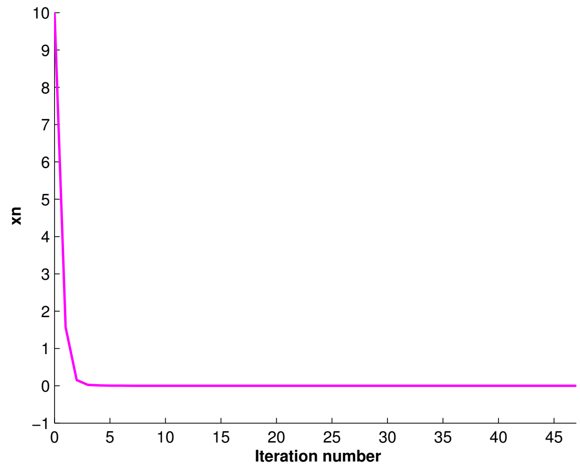

Figure 1: The iteration graph

In order to examine the influence of parameters involved in the algorithm (1.6), we further take the following set of parameters:

(i) $\alpha_n = \frac{1}{\sqrt{n+1}}$, $\beta_n = \frac{1}{(n+1)^2}$,

(ii) $\alpha_n = \frac{1}{\sqrt{2n+9}}$, $\beta_n = \frac{1}{(3n+2)^3}$,

(iii) $\alpha_n = \frac{1}{\sqrt{n+5}}$, $\beta_n = \frac{1}{(5n+2)^{5/2}}$,

(iv) $\alpha_n = \frac{1}{7n+1}$, $\beta_n = \frac{1}{(2n+4)^{3/2}}$, and

(v) $\alpha_n = \frac{1}{9n+5}$, $\beta_n = \frac{1}{(n+5)^7}$.

Using the above set of parameters, we now examine the influence of the number of members in the finite family. We set initial point $x_0 = 20$ and take $u_n = \frac{1}{(n+1)^3}$. The stopping criterion is $\|x_n - x^*\| \leq 10^{-5}$. The respective number of iterations for different values of N are reported in Table 2.

Table 2: The influence of number of members in the family

Parameters		Number of iterations for different values of N					
$\gamma_n = 1 - \alpha_n - \beta_n$		$N = 5$	$N = 10$	$N = 50$	$N = 100$	$N = 500$	$N = 1000$
$\alpha_n = \frac{1}{\sqrt{n+1}}$	$\beta_n = \frac{1}{(n+1)^2}$	49	49	49	49	49	49
$\alpha_n = \frac{1}{\sqrt{2n+9}}$	$\beta_n = \frac{1}{(3n+2)^3}$	48	48	48	48	48	48
$\alpha_n = \frac{1}{\sqrt{n+5}}$	$\beta_n = \frac{1}{(5n+2)^{5/2}}$	48	48	48	49	49	49
$\alpha_n = \frac{1}{7n+1}$	$\beta_n = \frac{1}{(2n+4)^{3/2}}$	46	46	46	46	46	46
$\alpha_n = \frac{1}{9n+5}$	$\beta_n = \frac{1}{(n+5)^7}$	46	46	46	46	46	46

Next, we tested the algorithm for different initial points and different set of parameters. The parameters N, u_n is fixed to 15 and $\frac{1}{(n+1)^3}$ respectively. Each iteration starts with a particular chosen x_0 and stops whenever $\|x_n - x^*\| \leq 10^{-5}$. Respective numbers of iterations are given in Table 3.

Table 3: The influence of initial point

Parameters $\gamma_n = 1 - \alpha_n - \beta_n$		Number of iterations for different initial points					
		$x_0 = -30$	$x_0 = -15$	$x_0 = -5$	$x_0 = 5$	$x_0 = 15$	$x_0 = 30$
$\alpha_n = \frac{1}{\sqrt{n+1}}$	$\beta_n = \frac{1}{(n+1)^2}$	49	49	49	49	49	49
$\alpha_n = \frac{1}{\sqrt{2n+9}}$	$\beta_n = \frac{1}{(3n+2)^3}$	48	48	48	48	48	48
$\alpha_n = \frac{1}{\sqrt{n+5}}$	$\beta_n = \frac{1}{(5n+2)^{5/2}}$	49	49	49	49	49	49
$\alpha_n = \frac{1}{7n+1}$	$\beta_n = \frac{1}{(2n+4)^{3/2}}$	47	47	47	47	47	47
$\alpha_n = \frac{1}{9n+5}$	$\beta_n = \frac{1}{(n+5)^7}$	47	47	47	47	47	47

Finally, test the algorithm for the set of α_n, β_n mentioned above and different error terms u_n. The parameters N, x_0 is fixed to 15 and 20 respectively. Each iteration stops whenever $\|x_n - x^*\| \leq 10^{-5}$. Respective numbers of iterations for different u_n are reported in Table 4.

Table 4: The influence of error term u_n

Parameters $\gamma_n = 1 - \alpha_n - \beta_n$		Number of iterations for different u_n				
		$u_n = \frac{1}{(5n+2)^3}$	$u_n = \frac{1}{(n+5)^4}$	$u_n = \frac{1}{(2n+1)^5}$	$u_n = \frac{1}{(5n+7)^6}$	$u_n = \frac{1}{(3n+5)^{3/2}}$
$\alpha_n = \frac{1}{\sqrt{n+1}}$	$\beta_n = \frac{1}{(n+1)^2}$	16	17	16	16	735
$\alpha_n = \frac{1}{\sqrt{2n+9}}$	$\beta_n = \frac{1}{(3n+2)^3}$	12	15	11	11	730
$\alpha_n = \frac{1}{\sqrt{n+5}}$	$\beta_n = \frac{1}{(5n+2)^{5/2}}$	14	16	14	14	735
$\alpha_n = \frac{1}{7n+1}$	$\beta_n = \frac{1}{(2n+4)^{3/2}}$	10	14	6	6	717
$\alpha_n = \frac{1}{9n+5}$	$\beta_n = \frac{1}{(n+5)^7}$	10	14	6	5	717

Computational results show that the new algorithm is quite efficient, stable and effective no matter, how many member is in the family of mappings or what initial point is chosen. However, Table 4 indicates that the error term u_n plays a significant role in the convergence.

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