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Convergence of general iteration scheme for total asymptotically nonexpansive mappings in CAT(0) spaces

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Abstract - The aim of this paper is to establish the Δ and strong convergence theorems of general iteration scheme for the class of total asymptotically nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and generalize some known results from previous works given in the existing literature.

Key words and phrases : Total asymptotically nonexpansive mapping, Δ-convergence, strong convergence, general iteration scheme, common fixed point, CAT(0) space.

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1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Fixed point theory in CAT(0) space has been first studied by Kirk (see [21, 22]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [3]).

Nanjaras and Panyanak [26] established the demiclosed principle for asymptotically nonexpansive mappings and gave the Δ-convergence theorem of the modified Mann iteration process for above mentioned mappings in a CAT(0) space. In 2010, Y. Niwongsa and B. Panyanak [27] studied the Noor iteration scheme in CAT(0) spaces and they proved some Δ and strong convergence theorems for asymptotically nonexpansive mappings which extend and improve some recent results from the literature. In 2013, Şahin and Başarir [28] studied the modified S-iteration process and established some strong convergence theorems under appropriate conditions for asymptotically quasi-nonexpansive mappings in the setting of CAT(0) spaces which generalize some results of Khan and Abbas [18]. Recently, the same authors in [2] studied and established Δ and strong convergence theorems of the modified S-iteration process for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces which generalize the result of Chang et al. [5] and Nanjaras and Panyanak [26].

Algorithm 1.1. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n) x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n) x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called modified Noor iterative sequence (see [30]).

If $\gamma_n = 0$ for all $n \geq 1$, then Algorithm 1.1 reduces to the following.

Algorithm 1.2. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$\begin{cases} y_n = \beta_n T^n x_n \oplus (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n) x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are appropriate sequences in [0,1] is called an Ishikawa iterative sequence (see [13], see also [24]).

If $\beta_n = 0$ for all $n \geq 1$, then Algorithm 1.2 reduces to the following.

Algorithm 1.3. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} \ = \ \alpha_n T^n x_n \oplus (1-\alpha_n) x_n, \quad n \geq 1, \tag{1.3}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1) is called a Mann iterative sequence (see [25], see also [24]).

Algorithm 1.4. The sequence $\{x_n\}$ defined by $x_1 \in K$ and

$$x_{n+1} = U_{n(r)} x_n, \ n \geq 1, \tag{1.4}$$

where

$$\begin{cases} U_{n(0)} = I, \text{the identity map}, \\ U_{n(1)} x = (1-\alpha_{n(1)}) x \oplus \alpha_{n(1)} T_1^n U_{n(0)} x, \\ \quad \vdots \\ U_{n(r)} x = (1-\alpha_{n(r)}) x \oplus \alpha_{n(r)} T_r^n U_{n(r-1)} x, \ n \geq 1, \end{cases}$$

where $0 \leq \alpha_{n(i)} \leq 1$ for each $i \in I$ is called a general iteration scheme (see [16]).

The iteration scheme (1.4) provides analogues of

(i) the scheme (1.3) if $r = 1$ and $T_1 = T$;

(ii) the scheme (1.2) if $r = 2$ and $T_1 = T_2 = T$ and

(iii) the scheme (1.1) if $r = 3$, $T_1 = T_2 = T_3 = T$.

In this paper, we study general iteration scheme (1.4) and establish some strong and Δ-convergence theorems for total asymptotically nonexpansive mappings in the framework of CAT(0) spaces. Our results extend and generalize some known results from previous works given in the existing literature.

2. Preliminaries notes and lemmas

Let (X, d) be a metric space and K be its nonempty subset. Let $T\colon K \to K$ be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$.

Definition 2.1. *Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ said to be*

(1) *nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in K$;*

(2) *asymptotically nonexpansive (see [9]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \le k_n\, d(x, y)$ for all $x, y \in K$ and $n \ge 1$;*

(3) *uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \le L\, d(x, y)$ for all $x, y \in K$ and $n \ge 1$;*

(4) *semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.*

Remark 2.1. From the above definitions, it is clear that each nonexpansive mapping is an asymptotically nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$, $\forall\, n \ge 1$ and an asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n\ge1}\{k_n\}$.

Chang et al. (see [5]) defined the concept of total asymptotically nonexpansive mapping as follows.

Definition 2.2. (see [5, Definition 2.1]) *Let (X, d) be a metric space, K be its nonempty subset and let $T\colon K \to K$ be a mapping. T is said to be a total asymptotically nonexpansive mapping if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that*

$$d(T^n x, T^n y) \le d(x, y) + \nu_n \psi(d(x, y)) + \mu_n$$

for all $x, y \in K$ and $n \ge 1$.

Remark 2.2. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\mu_n = 0$, $\nu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

We now give the definition and some basic properties of CAT(0) space.

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry, and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [3]).

CAT(0) space. A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \quad \leq \quad d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{2.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [15]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \quad \leq \quad \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{2.2}$$

The inequality (2.3) is the (CN) inequality of Bruhat and Tits [4]. The above inequality was extended in [6] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1 - \alpha) y) \quad \leq \quad & \alpha d^2(z, x) + (1 - \alpha) d^2(z, y) \\ & - \alpha(1 - \alpha) d^2(x, y) \end{aligned} \tag{2.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [3, p.163]). Moreover, if X is a $CAT(0)$

metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \quad \leq \quad \alpha d(z,x) + (1-\alpha)d(z,y), \tag{2.4}$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset K of a $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x,y] \subset K$.

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 2.1. (see [27]) *Let X be a $CAT(0)$ space.*

(i) *For $x,\, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,\, z) = t\, d(x,\, y) \quad \textit{and} \quad d(y,\, z) = (1-t)\, d(x,\, y). \quad \text{(A)}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x,\, y \in X$ and $t \in [0,1]$, we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a CAT(0) space X. For $x \in X$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \big\{x \in X : r(\{x_n\}) = r(x, \{x_n\})\big\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [7, Proposition 7].

We now recall the definition of Δ-convergence and weak convergence ($\rightharpoonup$) in CAT(0) space.

Definition 2.3. (see [23]) *A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.*

In this case we write $\Delta - \lim_n x_n = x$ and call x is the Δ-limit of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that, every bounded sequence has a regular subsequence [10, Lemma 15.2].

Since in a CAT(0) space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence, also it is noticed that [23, p.3690].

Lemma 2.2. (see [1]) *Given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, then*

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.4. (see [12]) *Let K be a closed convex subset of a CAT(0) space X. A bounded sequence $\{x_n\}$ in K is said to converge weakly to $q \in K$ if and only if $\Phi(q) = \inf_{x\in K} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.*

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_K\{x_n\} = \{q\}$.

Nanjaras and Panyanak (see [26]) established the following relation between Δ-convergence and weak convergence in a CAT(0) space.

Lemma 2.3. (see [26, Proposition 3.12]) *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X and let K be a closed convex subset of X which contains $\{x_n\}$. Then*

(i) Δ-$\lim_{x_n} = x$ *implies* $x_n \rightharpoonup x$.

(ii) *The converse of (i) is true if $\{x_n\}$ is regular.*

Lemma 2.4. (see [6, Lemma 2.8]) *If $\{x_n\}$ is a bounded sequence in a CAT(0) space X with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5. (see [8, Proposition 2.1]) *If K be a closed convex subset of a CAT(0) space X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.6. (see [16, Lemma 3.1]) *Let K be a nonempty bounded closed convex subset of a CAT(0) space X. Let $T_i : i \in I = \{1, 2, \ldots, r\}$ be a family of uniformly L-Lipschitzian self-maps on K. Then for $\{x_n\}$ in* (1.4) *with* $\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0$, *we have* $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ *for each $i \in I$.*

Lemma 2.7. (see [29]) *Suppose that* $\{a_n\}$, $\{b_n\}$ *and* $\{r_n\}$ *be sequences of nonnegative numbers such that* $a_{n+1} \leq (1+b_n)a_n + r_n$ *for all* $n \geq 1$. *If* $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.8. (see [5, Theorem 2.8]) *Let* K *be closed convex subset of a complete CAT(0) space* X *and let* $T\colon K \to K$ *be a total asymptotically nonexpansive and uniformly L-Lipschitzian mapping. Let* $\{x_n\}$ *be a bounded sequence in* K *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $\Delta - \lim_{n\to\infty} x_n = p$. *Then* $Tp = p$.

3. Main Results

In this section, we establish strong and Δ-convergence theorems using general iteration scheme (1.4) for total asymptotically nonexpansive mappings in the setting of CAT(0) spaces.

Lemma 3.1. *Let* K *be a nonempty closed convex subset of a complete CAT*(0) *space* X *and let* $T\colon K \to K$ *be a total asymptotically nonexpansive mapping with* $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. *Suppose that* $\{x_n\}$ *is defined by the iteration process* (1.4). *If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) *there exists a constant* $R > 0$ *such that* $\psi(t) \leq Rt$, $t \geq 0$.

Then $\lim_{n\to\infty} d(x_n, p)$ *and* $\lim_{n\to\infty} d(x_n, F)$ *exist for all* $p \in F$.

Proof. For any $p \in F$, from (1.4) and Lemma 2.1(ii), we have

$$\begin{aligned} d(U_{n(1)}x_n, p) &= d((1-\alpha_{n(1)})x_n \oplus \alpha_{n(1)}T_1^n x_n, p) \\ &\leq (1-\alpha_{n(1)})d(x_n, p) + \alpha_{n(1)}d(T_1^n x_n, p) \\ &\leq (1-\alpha_{n(1)})d(x_n, p) + \alpha_{n(1)}[d(x_n, p) + \nu_n\psi(d(x_n, p)) + \mu_n] \\ &\leq (1-\alpha_{n(1)})d(x_n, p) + \alpha_{n(1)}[d(x_n, p) + \nu_n R d(x_n, p) + \mu_n] \\ &= (1-\alpha_{n(1)})d(x_n, p) + \alpha_{n(1)}[(1+\nu_n R)d(x_n, p) + \mu_n] \\ &\leq (1+\nu_n R)d(x_n, p) + \mu_n. \end{aligned} \tag{3.1}$$

Again from (1.4), Lemma 2.1(ii) and using (3.1), we obtain

$$\begin{aligned} d(U_{n(2)}x_n, p) &= d((1-\alpha_{n(2)})x_n \oplus \alpha_{n(2)}T_2^n U_{n(1)}x_n, p) \\ &\leq (1-\alpha_{n(2)})d(x_n, p) + \alpha_{n(2)}d(T_2^n U_{n(1)}x_n, p) \\ &\leq (1-\alpha_{n(2)})d(x_n, p) + \alpha_{n(2)}[d(U_{n(1)}x_n, p) \\ &\quad +\nu_n\psi(d(U_{n(1)}x_n, p)) + \mu_n] \\ &\leq (1-\alpha_{n(2)})d(x_n, p) + \alpha_{n(2)}[d(U_{n(1)}x_n, p) \\ &\quad +\nu_n R d(U_{n(1)}x_n, p) + \mu_n] \end{aligned}$$

$$\begin{aligned}
&= (1-\alpha_{n(2)})d(x_n,p)+\alpha_{n(2)}[(1+\nu_nR)d(U_{n(1)}x_n,p)+\mu_n] \\
&\le (1-\alpha_{n(2)})d(x_n,p)+\alpha_{n(2)}[(1+\nu_nR)\{(1+\nu_nR)d(x_n,p)+\mu_n\} \\
&\quad +\mu_n] \\
&\le (1+\nu_nR)^2d(x_n,p)+(2+\nu_nR)\mu_n. \qquad (3.2)
\end{aligned}$$

Continuing the above process, using (1.4) and Lemma 2.1(ii), we get

$$\begin{aligned}
d(x_{n+1},p) &= d((1-\alpha_{n(r)})x_n \oplus \alpha_{n(r)}T_r^nU_{n(r-1)}x_n,p) \\
&\le (1-\alpha_{n(r)})d(x_n,p)+\alpha_{n(r)}d(T_r^nU_{n(r-1)}x_n,p) \\
&\le (1-\alpha_{n(r)})d(x_n,p)+\alpha_{n(r)}[d(U_{n(r-1)}x_n,p) \\
&\quad +\nu_n\psi(d(U_{n(r-1)}x_n,p))+\mu_n] \\
&\le (1-\alpha_{n(r)})d(x_n,p)+\alpha_{n(r)}[d(U_{n(r-1)}x_n,p)) \\
&\quad +\nu_nRd(U_{n(r-1)}x_n,p+\mu_n] \\
&= (1-\alpha_{n(r)})d(x_n,p)+\alpha_{n(r)}[(1+\nu_nR)d(U_{n(r-1)}x_n,p)+\mu_n] \\
&\le (1-\alpha_{n(r)})d(x_n,p)+\alpha_{n(r)}[(1+\nu_nR)\{(1+\nu_nR)^{r-1}d(x_n,p) \\
&\quad +((r-1)+\nu_nR)\mu_n\}+\mu_n] \\
&\le (1+\nu_nR)^rd(x_n,p)+(r+\nu_nR)\mu_n \\
&= (1+h_n)d(x_n,p)+f_n \qquad (3.3)
\end{aligned}$$

where $h_n = \binom{r}{1}\nu_nR+\binom{r}{2}(\nu_nR)^2+\cdots+\binom{r}{r}(\nu_nR)^r$ and $f_n=(r+\nu_nR)\mu_n$. Since by the hypothesis of the theorem $\sum_{n=1}^{\infty}\mu_n<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$, it follows that $\sum_{n=1}^{\infty}h_n<\infty$ and $\sum_{n=1}^{\infty}f_n<\infty$. Now equation (3.3) implies that

$$d(x_{n+1},F) \;\le\; (1+h_n)d(x_n,F)+f_n. \qquad (3.4)$$

It follows from equation (3.3), (3.4) and Lemma 2.7 that both $\lim_{n\to\infty}d(x_n,p)$ and $\lim_{n\to\infty}d(x_n,F)$ exist. This completes the proof. □

Theorem 3.1. *Let X, K, T, $\{x_n\}$ satisfy the hypothesis of Lemma* 3.1. *Assume that $F=\cap_{i=1}^{r}F(T_i)\neq\emptyset$. Then the sequence $\{x_n\}$ converges strongly to a point in F if and only if* $\liminf_{n\to\infty}d(x_n,F)=0$*, where* $d(x,F)=\inf\{d(x,p):p\in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty}d(x_n,F)=0$. As proved in Lemma 3.1, for all $p\in F$, $\lim_{n\to\infty}d(x_n,F)$ exists. Thus by hypothesis $\lim_{n\to\infty}d(x_n,F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. With the help of inequality $1+x\le e^x$, $x\ge 0$. For any integer $m\ge 1$, therefore from (3.3), we have

$$
\begin{aligned}
d(x_{n+m},p) &\leq (1+h_{n+m-1})d(x_{n+m-1},p)+f_{n+m-1} \\
&\leq e^{h_{n+m-1}}d(x_{n+m-1},p)+f_{n+m-1} \\
&\leq e^{h_{n+m-1}}[e^{h_{n+m-2}}d(x_{n+m-2},p)+f_{n+m-2}] \\
&\quad +f_{n+m-1} \\
&\leq e^{(h_{n+m-1}+h_{n+m-2})}d(x_{n+m-2},p) \\
&\quad +e^{h_{n+m-1}}[f_{n+m-2}+f_{n+m-1}] \\
&\leq \dots \\
&\leq \left(e^{\sum_{k=n}^{n+m-1}h_k}\right)d(x_n,p)+\left(e^{\sum_{k=n}^{n+m-1}h_k}\right)\sum_{k=n}^{n+m-1}f_k \\
&\leq \left(e^{\sum_{k=n}^{\infty}h_k}\right)d(x_n,p)+\left(e^{\sum_{k=n}^{\infty}h_k}\right)\sum_{k=n}^{n+m-1}f_k \\
&= Q\,d(x_n,p)+Q\sum_{k=n}^{n+m-1}f_k \qquad (3.5)
\end{aligned}
$$

where $Q=e^{\sum_{k=n}^{\infty}h_k}$. Since $\lim_{n\to\infty}d(x_n,F)=0$, without loss of generality, we may assume that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_{n_k}\}\subset F$ such that $d(x_{n_k},p_{n_k})\to 0$ as $k\to\infty$. Then for any $\varepsilon>0$, there exists $k_\varepsilon>0$ such that

$$
d(x_{n_k},p_{n_k})<\frac{\varepsilon}{4Q} \quad \text{and} \quad \sum_{k=n_{k_\varepsilon}}^{\infty}f_k<\frac{\varepsilon}{4Q}, \qquad (3.6)
$$

for all $k\geq k_\varepsilon$.

For any $m\geq 1$ and for all $n\geq n_{k_\varepsilon}$, by (3.5) and (3.6), we have

$$
\begin{aligned}
d(x_{n+m},x_n) &\leq d(x_{n+m},p_{n_k})+d(x_n,p_{n_k}) \\
&\leq Q\,d(x_n,p_{n_k})+Q\sum_{k=n_{k_\varepsilon}}^{\infty}f_k \\
&\quad +Q\,d(x_n,p_{n_k})+Q\sum_{k=n_{k_\varepsilon}}^{\infty}f_k \\
&= 2Q\,d(x_n,p_{n_k})+2Q\sum_{k=n_{k_\varepsilon}}^{\infty}f_k \\
&< 2Q.\frac{\varepsilon}{4Q}+2Q.\frac{\varepsilon}{4Q}=\varepsilon. \qquad (3.7)
\end{aligned}
$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty}x_n=q$.

Since K is closed, therefore $q \in K$. Next, we show that $q \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ we get $d(q, F) = 0$, closedness of F gives that $q \in F$. Thus $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Lemma 3.2. *Let X be a complete CAT(0) space and let K be a nonempty bounded closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian and total asymptotically nonexpansive self-maps on K. Suppose that $x_0 \in K$ with $0 < \delta \leq \alpha_{n(i)} < 1 - \delta$ for each $i \in I$ and for some $\delta \in (0, \frac{1}{2})$. Let $\{x_n\}$ be the sequence defined by (1.4). Assume that $F = \cap_{i=1}^{r} F(T_i) \neq \emptyset$. If the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) *there exists a constant $R > 0$ such that $\psi(t) \leq Rt$, $t \geq 0$.*

Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i \in I$.

Proof. Let $p \in F$ and apply the inequality (2.3) to the scheme (1.4) to get

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2((1-\alpha_{n(r)})x_n \oplus \alpha_{n(r)} T_r^n U_{n(r-1)} x_n, p) \\
&\leq (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} d^2(T_r^n U_{n(r-1)} x_n, p) \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \\
&\leq (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} [d(U_{n(r-1)} x_n, p) \\
&\quad +\nu_n \psi(d(U_{n(r-1)} x_n, p)) + \mu_n]^2 \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \\
&\leq (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} [d(U_{n(r-1)} x_n, p) \\
&\quad +\nu_n R d(U_{n(r-1)} x_n, p) + \mu_n]^2 \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \\
&= (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} [d(U_{n(r-1)} x_n, p) \\
&\quad +\nu_n R d(U_{n(r-1)} x_n, p) + \mu_n]^2 \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \\
&= (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} [d(U_{n(r-1)} x_n, p) + \theta_n]^2 \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \\
&\leq (1-\alpha_{n(r)}) d^2(x_n, p) + \alpha_{n(r)} d^2(U_{n(r-1)} x_n, p) + \alpha_{n(r)} S_{rn} \\
&\quad -\alpha_{n(r)}(1-\alpha_{n(r)}) d^2(x_n, T_r^n U_{n(r-1)} x_n) \qquad (3.8)
\end{aligned}$$

where $\theta_n = \nu_n R d(U_{n(r-1)} x_n, p) + \mu_n$ and $S_{rn} = 2d(U_{n(r-1)} x_n, p)\theta_n + \theta_n^2$. Since by assumption of the theorem $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$, it follows that $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\sum_{n=1}^{\infty} S_{rn} < \infty$. Again from (3.8), (1.4) and

(2.3), we get

$$
\begin{aligned}
d^2(x_{n+1},p) \leq\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}d^2(U_{n(r-1)}x_n,p)+\alpha_{n(r)}S_{rn} \\
& -\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
=\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}d^2((1-\alpha_{n(r-1)})x_n\oplus\alpha_{n(r-1)} \\
& \times T_{r-1}^nU_{n(r-2)}x_n,p)+\alpha_{n(r)}S_{rn} \\
& -\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
\leq\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}[(1-\alpha_{n(r-1)})d^2(x_n,p) \\
& +\alpha_{n(r-1)}\times d^2(T_{r-1}^nU_{n(r-2)}x_n,p) \\
& -\alpha_{n(r-1)}(1-\alpha_{n(r-1)})d^2(x_n,T_{r-1}^nU_{n(r-2)}x_n)] \\
& +\alpha_{n(r)}S_{rn}-\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
\leq\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}(1-\alpha_{n(r-1)})d^2(x_n,p) \\
& +\alpha_{n(r)}\alpha_{n(r-1)}\times[d(U_{n(r-2)}x_n,p)+\nu_n\psi(d(U_{n(r-2)}x_n,p)) \\
& +\mu_n]^2-\alpha_{n(r)}\alpha_{n(r-1)}(1-\alpha_{n(r-1)})\times d^2(x_n,T_{r-1}^nU_{n(r-2)}x_n) \\
& +\alpha_{n(r)}S_{rn}-\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
\leq\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}(1-\alpha_{n(r-1)})d^2(x_n,p)+ \\
& \alpha_{n(r)}\alpha_{n(r-1)}\times[d(U_{n(r-2)}x_n,p)+\nu_nRd(U_{n(r-2)}x_n,p)+\mu_n]^2 \\
& -\alpha_{n(r)}\alpha_{n(r-1)}(1-\alpha_{n(r-1)})\times d^2(x_n,T_{r-1}^nU_{n(r-2)}x_n) \\
& +\alpha_{n(r)}S_{rn}-\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
=\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}(1-\alpha_{n(r-1)})d^2(x_n,p) \\
& +\alpha_{n(r)}\alpha_{n(r-1)}\times[d(U_{n(r-2)}x_n,p)+\lambda_n]^2 \\
& -\alpha_{n(r)}\alpha_{n(r-1)}(1-\alpha_{n(r-1)})\times d^2(x_n,T_{r-1}^nU_{n(r-2)}x_n) \\
& +\alpha_{n(r)}S_{rn}-\alpha_{n(r)}(1-\alpha_{n(r)})d^2(x_n,T_r^nU_{n(r-1)}x_n) \\
=\ & (1-\alpha_{n(r)})d^2(x_n,p)+\alpha_{n(r)}(1-\alpha_{n(r-1)})d^2(x_n,p) \\
& +\alpha_{n(r)}\alpha_{n(r-1)}\times d^2(U_{n(r-2)}x_n,p)+\alpha_{n(r)}S_{rn} \\
& +\alpha_{n(r)}\alpha_{n(r-1)}S_{(r-1)n}-\alpha_{n(r)}(1-\alpha_{n(r)})\times \\
& d^2(x_n,T_r^nU_{n(r-1)}x_n)-\alpha_{n(r)}\alpha_{n(r-1)}(1-\alpha_{n(r-1)})\times \\
& d^2(x_n,T_{r-1}^nU_{n(r-2)}x_n) \qquad (3.9)
\end{aligned}
$$

where $\lambda_n=\nu_nRd(U_{n(r-2)}x_n,p)+\mu_n$ and $S_{(r-1)n}=2d(U_{n(r-2)}x_n,p)\lambda_n+\lambda_n^2$. Since by the assumption of the theorem $\sum_{n=1}^{\infty}\nu_n<\infty$ and $\sum_{n=1}^{\infty}\mu_n<\infty$, it follows that $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\sum_{n=1}^{\infty}S_{(r-1)n}<\infty$.

After applying the inequality (2.3) to the scheme (1.4) r times, we get

$$\begin{aligned}
d^2(x_{n+1},p) \leq\ & \Big[\prod_{i=1}^{r}\alpha_{n(i)}+\Big\{\prod_{i=2}^{r}\alpha_{n(i)}-\prod_{i=1}^{r}\alpha_{n(i)}\Big\} \\
&+\Big\{\prod_{i=3}^{r}\alpha_{n(i)}-\prod_{i=2}^{r}\alpha_{n(i)}\Big\}+\dots \\
&+\{\alpha_{n(r)}-\alpha_{n(r)}\alpha_{n(r-1)}\}\Big]d^2(x_n,p) \\
&+\Big(\prod_{i=1}^{r}\alpha_{n(i)}\Big)S_{1n}+\Big(\prod_{i=2}^{r}\alpha_{n(i)}\Big)S_{2n}+\dots+\alpha_{n(r)}S_{rn} \\
&-(1-\alpha_{n(1)})\prod_{i=1}^{r}\alpha_{n(i)}d^2(x_n,T_1^n x_n) \\
&-(1-\alpha_{n(2)})\prod_{i=2}^{r}\alpha_{n(i)}d^2(x_n,T_2^n U_{n(1)}x_n) \\
&\vdots \\
&-(1-\alpha_{n(r)})\alpha_{n(r)}d^2(x_n,T_r^n U_{n(r-1)}x_n) \\
\leq\ & \Big[\prod_{i=1}^{r}\alpha_{n(i)}+\Big\{\prod_{i=2}^{r}\alpha_{n(i)}-\prod_{i=1}^{r}\alpha_{n(i)}\Big\} \\
&+\Big\{\prod_{i=3}^{r}\alpha_{n(i)}-\prod_{i=2}^{r}\alpha_{n(i)}\Big\}+\dots \\
&+\{\alpha_{n(r)}-\alpha_{n(r)}\alpha_{n(r-1)}\}\Big]d^2(x_n,p)+\Big(\prod_{i=1}^{r}\alpha_{n(i)}\Big)\sum_{j=1}^{r}S_{jn} \\
&-(1-\alpha_{n(1)})\prod_{i=1}^{r}\alpha_{n(i)}d^2(x_n,T_1^n x_n) \\
&-(1-\alpha_{n(2)})\prod_{i=2}^{r}\alpha_{n(i)}d^2(x_n,T_2^n U_{n(1)}x_n) \\
&\vdots \\
&-(1-\alpha_{n(r)})\alpha_{n(r)}d^2(x_n,T_r^n U_{n(r-1)}x_n) \\
\leq\ & \Big[\prod_{i=1}^{r}\alpha_{n(i)}+\Big\{\prod_{i=2}^{r}\alpha_{n(i)}-\prod_{i=1}^{r}\alpha_{n(i)}\Big\} \\
&+\Big\{\prod_{i=3}^{r}\alpha_{n(i)}-\prod_{i=2}^{r}\alpha_{n(i)}\Big\}+\dots \\
&+\{\alpha_{n(r)}-\alpha_{n(r)}\alpha_{n(r-1)}\}\Big]d^2(x_n,p)+M\sum_{j=1}^{r}S_{jn}
\end{aligned}$$

$$
\begin{aligned}
&-(1-\alpha_{n(1)})\prod_{i=1}^{r}\alpha_{n(i)}d^2(x_n, T_1^n x_n) \\
&-(1-\alpha_{n(2)})\prod_{i=2}^{r}\alpha_{n(i)}d^2(x_n, T_2^n U_{n(1)}x_n) \\
&\vdots \\
&-(1-\alpha_{n(r)})\alpha_{n(r)}d^2(x_n, T_r^n U_{n(r-1)}x_n) \qquad (3.10)
\end{aligned}
$$

where $M = \left(\prod_{i=1}^{r}\alpha_{n(i)}\right) > 0$. From the above computations, we have the following r inequalities

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq d^2(x_n,p) + MS_{1n} \\
&\quad -(1-\alpha_{n(1)})\prod_{i=1}^{r}\alpha_{n(i)}d^2(x_n, T_1^n x_n) \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq d^2(x_n,p) + MS_{2n} \\
&\quad -(1-\alpha_{n(2)})\prod_{i=2}^{r}\alpha_{n(i)}d^2(x_n, T_2^n U_{n(1)}x_n) \qquad (3.12)
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq d^2(x_n,p) + MS_{(r-1)n} - \alpha_{n(r)}\alpha_{n(r-1)}(1-\alpha_{n(r-1)}) \times \\
&\quad d^2(x_n, T_{r-1}^n U_{n(r-2)}x_n) \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq d^2(x_n,p) + MS_{rn} - \alpha_{n(r)}(1-\alpha_{n(r)}) \times \\
&\quad d^2(x_n, T_r^n U_{n(r-1)}x_n). \qquad (3.14)
\end{aligned}
$$

Using $0 < \delta \leq \alpha_{n(i)} < 1-\delta$ for each $i \in I$ and $\lim_{n\to\infty} S_{in} = 0$ for each $i \in I$ in the above inequalities and then simplifying, we have

$$\delta^{r+1}d^2(x_n, T_1^n x_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) \qquad (3.15)$$

$$\delta^{r}d^2(x_n, T_2^n U_{n(1)}x_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) \qquad (3.16)$$

$$\vdots$$

$$\delta^{2}d^2(x_n, T_r^n U_{n(r-1)}x_n) \leq d^2(x_n,p) - d^2(x_{n+1},p). \qquad (3.17)$$

The sequence $\{d(x_n, p)\}$ is convergent, therefore from the inequalities (3.15)-(3.17), we deduce

$$\lim_{n\to\infty} d(x_n, T_i^n U_{n(i-1)} x_n) = 0 \quad \text{for } i \in I. \tag{3.18}$$

Further,

$$\begin{aligned} d(x_n, T_2^n x_n) &\leq d(x_n, T_2^n U_{n(1)} x_n) + d(T_2^n U_{n(1)} x_n, T_2^n x_n) \\ &\leq d(x_n, T_2^n U_{n(1)} x_n) + L\, d(U_{n(1)} x_n, x_n) \\ &= d(x_n, T_2^n U_{n(1)} x_n) + L\, d((1-\alpha_{n(1)}) x_n \oplus \alpha_{n(1)} T_1^n x_n, x_n) \\ &\leq d(x_n, T_2^n U_{n(1)} x_n) + L\, \alpha_{n(1)} d(T_1^n x_n, x_n) \end{aligned}$$

together with (3.17) (for $i = 2$) gives that

$$\lim_{n\to\infty} d(x_n, T_2^n x_n) = 0. \tag{3.19}$$

By similar methods show that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0 \quad \text{for each } i \in I. \tag{3.20}$$

Finally, by Lemma 2.6, we get

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 \quad \text{for each } i \in I.$$

This completes the proof. □

Now we are in a position to prove the Δ-convergence and strong convergence results.

Theorem 3.2. *Let X be a complete CAT(0) space and K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian total asymptotically nonexpansive self-maps on K. Suppose that $x_0 \in K$ with $0 < \delta \leq \alpha_{n(i)} < 1-\delta$ for each $i \in I$ and for some $\delta \in (0, \frac{1}{2})$. Let $\{x_n\}$ be the sequence defined by (1.4). Assume that $F = \cap_{i=1}^r F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ Δ-converges to a point of F.*

Proof. Let $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We can complete the proof by showing that $\omega_w(x_n) \subseteq F$ and $\omega_w(x_n)$ consists of exactly one point. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v \in K$. By Lemma 2.8, $v \in F(T_l)$. By arbitrariness of $l \in \{1, 2, \dots, r\}$, we have $v \in F$. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, v)$ exists, so by Lemma 2.4, $v = u$, i.e., $\omega_w(x_n) \subseteq F$.

To show that $\{x_n\}$ Δ-converges to a point in F, it is sufficient to show that $\omega_w(x_n)$ consists of exactly one point.

Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$ and let $A(\{x_n\}) = \{x\}$. Since $w \in \omega_w(x_n) \subseteq F$ and by Lemma 3.1, $\lim_{n\to\infty} d(x_n, w)$ exists. Again by Lemma 2.4, we have $x = w \in F$. Thus $\omega_w(x_n) = \{x\}$. This shows that $\{x_n\}$ Δ-converges to a point in F. This completes the proof. □

Theorem 3.3. *Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian total asymptotically nonexpansive self-maps on K. Suppose that $x_0 \in K$ with $0 < \delta \leq \alpha_{n(i)} < 1-\delta$ for each $i \in I$ and for some $\delta \in (0, \frac{1}{2})$. Let $\{x_n\}$ be the sequence defined by (1.4). Assume that $F = \cap_{i=1}^r F(T_i) \neq \emptyset$. If one of the mappings in $\{T_1, T_2, \ldots, T_r\}$ is semi-compact, then the sequence $\{x_n\}$ converges strongly to a point in F.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, \ldots, r\}$. By Lemma 3.2, $\lim_{n\to\infty} d(x_n, T_{i_0} x_n) = 0$ for each $i \in I$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $p \in K$ such that $\lim_{j\to\infty} x_{n_j} = p$. Now Lemma 3.2 guarantees that $\lim_{n_j\to\infty} d(x_{n_j}, T_l x_{n_j}) = 0$ for all $l \in \{1, 2, \ldots, r\}$ and so $d(p, T_l p) = 0$ for all $l \in \{1, 2, \ldots, r\}$. This implies that $p \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, it follows as in the proof of Theorem 3.1 that the sequence $\{x_n\}$ converges strongly to a point in F. □

For our next result, we need the following definition.

Definition 3.1. (see [16]) *A family of self-maps $\{T_i : i \in I = \{1, 2, \ldots, r\}$ on a subset K of a metric space (X, d) with at least one common fixed point is said to satisfy Condition (AV)* (see [16]) *if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $f(d(x, F)) \leq \frac{1}{r}\sum_{i=1}^r d(x, T_i x)$ for all $x \in K$.*

As an application of Theorem 3.1, we establish another strong convergence result employing Condition (AV).

Theorem 3.4. *Let X be a complete $CAT(0)$ space and let K be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of uniformly L-Lipschitzian total asymptotically nonexpansive self-maps on K. Assume that $F = \cap_{i=1}^r F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the general iteration scheme defined by (1.4) with $0 < \delta \leq \alpha_{n(i)} < 1 - \delta$ for each $i \in I$ and for some $\delta \in (0, \frac{1}{2})$. If $\{T_i : i \in I\}$ satisfies Condition (AV), then the sequence $\{x_n\}$ converges strongly to a point in F.*

Proof. As in the proof of Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Again by Lemma 3.2 we have $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for $l = 1, 2, \ldots, r$. So Condition (AV) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 3.1 implies that $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Remark 3.1. Theorem 3.2 extends Theorem 5.7 of Nanjaras and Panyanak (see [26]) and Theorem 3.5 of Niwongsa and Panyanak (see [27]) to the case of more general class of asymptotically nonexpansive mapping and general iteration scheme defined by (1.4) considered in this paper.

Remark 3.2. Theorem 3.2 also extends Theorem 17 of Karapinar et al. in (see [14]) to the case of finite family of total asymptotically nonexpansive mappings and general iteration scheme considered in this paper.

Remark 3.3. Our results also extend and generalize the corresponding results of Başarir and Şahin (see [2]) to the case of finite family of total asymptotically nonexpansive mappings and general iteration scheme considered in this paper.

Remark 3.4. Our results also extend and generalize several known results from the current existing literature.

Example 3.1. (See [11], Example 3.1) Let $\mathbb{R}$ be the real line with the usual norm $\|.\|$ and $K = [-1, 1]$. Define a mapping $T\colon K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0). \end{cases}$$

Then T is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and uniformly L-Lipschtzian mapping with $L = \sup_{n\geq 1}\{k_n\}$ and hence it is a total asymptotically nonexpansive mapping by Remark 2.2. Also the fixed point of T, that is, $F(T) = \{0\}$.

4. Conclusion

In this paper, we establish some Δ and strong convergence theorems using general iteration scheme (1.4) for more general class of asymptotically nonexpansive mappings in the framework of CAT(0) spaces and also give its application. The results presented in this paper extend and generalize some known results from the current existing literature (see, for example, [2, 14, 26, 27, 30] and many others).

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