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Notes and comments on sufficient first-order optimality conditions in scalar and vector optimization

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Abstract - We present several notes and comments regarding sufficient first-order optimality conditions for constrained optimization problems. In the first half of the paper scalar nonlinear programming problems are considered; in the second half of the paper Pareto optimization problems are considered.

Key words and phrases : Sufficient optimality conditions, generalized convex functions, scalar optimization, vector optimization.

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1. Introduction

There is a vast literature concerning sufficient first-order optimality conditions for scalar and vector optimization problems. The related results have grown up together with the development of the studies on various types of generalized convexity and generalized monotonicity. However, we have often noticed an "uncontrolled growth" of the various classes of generalized convex functions. Giannessi asserts in [16] that "unfortunately many generalizations look like mere formal mathematics without any motivation and contribute to drive mathematics away from the real world". In this paper we shall point out some remarks concerning sufficient optimality conditions of the Karush-Kuhn-Tucker type and of the Fritz John type, for scalar optimization problems and for vector optimization problems. We shall exploit the most used (and useful) generalized convex functions proposed for these kinds of problems.

Definition 1.1. *A differentiable scalar function $f : A \longrightarrow \mathbb{R}$, $A \subset \mathbb{R}^n$ open convex set, is:*

- ***convex*** [cx] *on A if*

$$f(x) - f(x^0) \geqq \nabla f(x^0)(x - x^0), \quad \forall x, x^0 \in A.$$

- ***strictly convex*** [scx] *on* A *if*

$$f(x) - f(x^0) > \nabla f(x^0)(x - x^0), \quad \forall x, x^0 \in A, x \neq x^0.$$

- ***quasiconvex*** [qcx] *on* A *if*

$$f(x) \leqq f(x^0) \Longrightarrow \nabla f(x^0)(x - x^0) \leqq 0, \quad \forall x, x^0 \in A,$$

i. e.

$$\nabla f(x^0)(x - x^0) > 0 \Longrightarrow f(x) > f(x^0), \quad \forall x, x^0 \in A.$$

- ***pseudoconvex*** [pcx] *on* A *if*

$$f(x) < f(x^0) \Longrightarrow \nabla f(x^0)(x - x^0) < 0, \quad \forall x, x^0 \in A,$$

i. e.

$$\nabla f(x^0)(x - x^0) \geqq 0 \Longrightarrow f(x) \geqq f(x^0), \quad \forall x, x^0 \in A.$$

- ***strictly pseudoconvex*** [spcx] *on* A *if*

$$f(x) \leqq f(x^0) \Longrightarrow \nabla f(x^0)(x - x^0) < 0, \quad \forall x, x^0 \in A, x \neq x^0,$$

i. e.

$$\nabla f(x^0)(x - x^0) \geqq 0 \Longrightarrow f(x) > f(x^0), \quad \forall x, x^0 \in A, x \neq x^0.$$

The following inclusion relations hold.

$$\begin{array}{ccc} & \mathsf{qcx} & \\ \Uparrow & & \Uparrow \\ \mathsf{spcx} & \Rightarrow & \mathsf{pcx} \\ \Uparrow & & \Uparrow \\ \mathsf{scx} & \Rightarrow & \mathsf{cx} \end{array}$$

2. Sufficient first-order optimality conditions in scalar optimization

2.1. The role of non-vanishing gradients in quasiconvex optimization

In making some comments on the paper [27] of Lasserre, the present author has focalized the role of non-vanishing gradients in a quasiconvex nonlinear programming problem: see [19]. In particular, the following result is recalled.

Theorem 2.1. *Assume that $A \subset \mathbb{R}^n$ is an open convex set, $f : A \longrightarrow \mathbb{R}$ is differentiable and $\nabla f(x) \neq 0$ for all $x \in A$. Then f is pseudoconvex on A if and only if f is quasiconvex on A.*

The above theorem is a consequence of a result of Crouzeix and Ferland (see [15]), stating that a differentiable and quasiconvex function on the open and convex set $A \subset \mathbb{R}^n$ is pseudoconvex on A if and only if f has a minimum at x whenever $\nabla f(x) = 0$. The proof of Crouzeix and Ferland of this last result is a bit complicate. Theorem 2.1 was also proved by Giorgi (see [17, 18]) and by Cambini and Martein (see [8]). We now present another simple proof.

Proof of Theorem 2.1. It is well-known that if f is pseudoconvex on A, then f is quasiconvex on A. Suppose now that f is quasiconvex on A (open and convex set of $\mathbb{R}^n$) and that $\nabla f(x) \neq 0$ for all $x \in A$. To prove that f is pseudoconvex on A we assume that $\nabla f(x^0)(x - x^0) \geqq 0$, for every $x, x^0 \in A$, and prove that this inequality implies $f(x) \geqq f(x^0)$. If $\nabla f(x^0)(x - x^0) > 0$ for some $x, x^0 \in A$ $(x \neq x^0)$ then, being f quasiconvex, we get $f(x) > f(x^0)$ and there is nothing other to prove. We need therefore to rule out the case $\nabla f(x^0)(x - x^0) = 0 \Longrightarrow f(x) < f(x^0)$. Assume, absurdly, that the said case occurs. We will show that, under the assumptions of the theorem, we can perturb x to x' so that

$$\nabla f(x^0)(x' - x^0) > 0 \Longrightarrow f(x') < f(x^0), \tag{2.1}$$

in contradiction to the assumption that f is quasiconvex. Let v be the nonzero vector $\nabla f(x^0)$. For $t > 0$ we have

$$\begin{aligned} \nabla f(x^0)(x + tv - x^0) &= \nabla f(x^0)(tv + x - x^0) = t\nabla f(x^0)v + \nabla f(x^0)(x - x^0) \\ &= t(\| \nabla f(x^0) \|)^2 + 0 > 0. \end{aligned}$$

Since f is continuous at x, we can choose t small enough so that $f(x + tv) < f(x^0)$ and $\nabla f(x^0)(x + tv - x^0) > 0$, that is $x' = x + tv$ satisfies (2.1), in contradiction with the assumed quasiconvexity of f. So, the case $\nabla f(x^0)(x - x^0) = 0 \Longrightarrow f(x) < f(x^0)$ cannot hold and therefore f is pseudoconvex. □

We have to note that the assumption on the non-vanishing gradient is essential for the validity of Theorem 2.1. For example, the function $\varphi(t) = t^3$, $t \in (-\infty, +\infty)$ is quasiconvex (and also quasiconcave), but not pseudoconvex (indeed $\varphi'(0) = 0$).

Following Mangasarian (see [30]), it is possible to give also "point-wise" definitions of convex and generalized convex functions at a fixed point $x^0 \in A$. For example, we accept the following definitions.

Definition 2.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable on some open set containing the set $A \subset \mathbb{R}^n$ (A not necessarily convex); f is* ***pseudoconvex*** *(respectively* ***quasiconvex****) at $x^0 \in A$, with respect to A, if*

$$\nabla f(x^0)(x - x^0) \geqq 0 \Longrightarrow f(x) \geqq f(x^0), \quad \forall x \in A.$$

$$(f(x) \leqq f(x^0) \Longrightarrow \nabla f(x^0)(x - x^0) \leqq 0, \quad \forall x \in A).$$

Theorem 2.1 can be restated in the following form.

Theorem 2.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable on some open set containing the set $A \subset \mathbb{R}^n$ and let f be quasiconvex at $x^0 \in A$, with respect to A; let be $\nabla f(x^0) \neq 0$. Then f is pseudoconvex at x^0 (with respect to A).*

Contrary to the "global" definitions, it is no longer true that if f is pseudoconvex at $x^0 \in A$, then f is quasiconvex at $x^0 \in A$. The following counterexample is taken from [32]. Consider the function $\varphi(t) = t - t^2$, $A = \{0, 1\}$, a two-points set. We have $\varphi(t) = 0$ for all $t \in A$, so that φ is pseudoconvex at $\bar{t} = 0$ by default. But $\varphi'(0)(1 - 0) = 1$, so that $\varphi(t)$ is not quasiconvex at $\bar{t} = 0$, with respect to A. Martos proves in [32] that we have the usual implication if A is a convex set.

Consider now the following basic nonlinear programming problem, with only inequality constraints.

$$(\mathsf{P}) \qquad \begin{cases} \text{minimize } f(x) \\ \text{subject to: } g_i(x) \leqq 0, \quad i = 1, ..., m, \end{cases}$$

where $f : A \longrightarrow \mathbb{R}$ and every $g_i : A \longrightarrow \mathbb{R}$ are differentiable on the open set $A \subset \mathbb{R}^n$. Denote by K the *feasible set* of (P) and denote by $I(x^0)$ the *set of the active constraints* at $x^0 \in K$, i.e.

$$I(x^0) = \left\{ i : g_i(x^0) = 0 \right\}.$$

It is well-known that if $x^0 \in K$ is a (local) solution of (P) and if a constraint qualification is satisfied, then the following Karush-Kuhn-Tucker ((KKT)) conditions hold: there exist multipliers $\lambda_1, ..., \lambda_m$ such that

$$\nabla f(x^0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^0) = 0; \tag{2.2}$$

$$\lambda_i g_i(x^0) = 0, \quad i = 1, ..., m; \tag{2.3}$$

$$\lambda_i \geqq 0, \quad i = 1, ..., m. \tag{2.4}$$

It is also well-known that the (KKT) conditions are sufficient for the optimality of $x^0 \in K$ (i.e. (2.2)-(2.3)-(2.4) are sufficient for $x^0 \in K$ to be solution of (P)) if either:

1) f is convex, every g_i, $i \in I(x^0)$, is convex.

2) f is pseudoconvex, every g_i, $i \in I(x^0)$, is quasiconvex.

3) f is pseudoconvex and $\sum_{i=1}^{m} \lambda_i g_i$ is quasiconvex (where λ_i is the i-th multiplier appearing in (2.2)-(2.3)-(2.4)).

4) The "restricted Lagrangian function" (of vector x only) $f + \sum_{i=1}^{m} \lambda_i g_i$, $\lambda_i \geqq 0, i = 1, ..., m$, is pseudoconvex.

We note that, unlike convex functions, pseudoconvex and quasiconvex functions are in general not additive. It is easy to prove the following result.

Theorem 2.3. *Let $x^0 \in K$ satisfy conditions* (2.2)-(2.3)-(2.4). *If either f is pseudoconvex at x^0 (with respect to A) or f is quasiconvex at x^0 (with respect to A) and it holds $\nabla f(x^0) \neq 0$, if every g_i, $i \in I(x^0)$, is quasiconvex at x^0 (with respect to A), then x^0 solves* (P).

Theorem 2.3 removes some unnecessary assumptions of the famous sufficient optimality conditions given in the pioneering paper [2] of Arrow and Enthoven. Moreover, on the ground of what previously expounded, its proof is quite straightforward (contrary to the original proof of Arrow and Enthoven).

2.2. Various first-order sufficient Fritz John and (KKT) optimality conditions for a problem with both inequality and equality constraints

Consider the following nonlinear programming problem with both inequality and equality constraints:

$$(\mathsf{P}_1) \qquad \begin{cases} \text{minimize } f(x) \\ \text{subject to: } g_i(x) \leqq 0, \quad i = 1, ..., m, \\ \qquad\qquad\quad h_j(x) = 0, \quad j = 1, ..., p, \end{cases}$$

where $f : A \longrightarrow \mathbb{R}$, and every $g_i : A \longrightarrow \mathbb{R}$, $i = 1, ...m$, are differentiable on the open set $A \subset \mathbb{R}^n$ and every $h_j : A \longrightarrow \mathbb{R}$, $j = 1, ..., p$, is continuously differentiable on A. We denote by K_1 the feasible set of (P_1).

The following necessary optimality conditions for (P_1) are well-known.

Theorem 2.4. (Fritz John) *If $x^0 \in K_1$ is a local solution for (P_1), then there exist $\alpha \geqq 0$, $u_i \geqq 0$, $i = 1, ..., m$, $v_j \in \mathbb{R}$, $j = 1, ..., p$, not all equal to zero, such that* ((FJ) *conditions*):

$$\alpha \nabla f(x^0) + \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} v_j \nabla h_j(x^0) = 0; \tag{2.5}$$

$$u_i g_i(x^0) = 0, \quad \forall i = 1, ..., m. \tag{2.6}$$

Theorem 2.5. (Karush-Kuhn-Tucker) *If $x^0 \in K_1$ is a local solution for* (P_1) *and if a constraint qualification holds, then in the previous* (FJ) *conditions we have $\alpha = 1$, i. e. there exist $u_i \geqq 0$, $i = 1, ..., m$, $v_j \in \mathbb{R}$, $j = 1, ..., p$, such that* ((KKT) *conditions*):

$$\nabla f(x^0) + \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} v_j \nabla h_j(x^0) = 0; \tag{2.7}$$

$$u_i g_i(x^0) = 0, \quad \forall i = 1, ..., m. \tag{2.8}$$

Now let us consider the function

$$\mathcal{L}(x) = \alpha f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} v_j h_j(x),$$

$\alpha \geqq 0$, $u_i \geqq 0$, $i = 1, ..., m$, $v_j \in \mathbb{R}$, $j = 1, ..., p$, (α, u_i, v_j) not all zero, function that we may call "restricted Fritz John-Lagrange function". The following results collect some sufficient first-order optimality conditions for (P_1).

Theorem 2.6. *Let $x^0 \in K_1$.*

(i) *If $\mathcal{L}$ is strictly convex, then* (FJ) *conditions are sufficient for x^0 to be solution of* (P_1).

(ii) *If $\mathcal{L}$ is convex, then* (KKT) *conditions are sufficient for x^0 to be solution of* (P_1).

(iii) *If f and g_i, $i = 1, ..., m$, are convex and h_j $j = 1, ..., p$, is linear, then* (KKT) *conditions are sufficient for x^0 to be solution of* (P_1).

Theorem 2.7. *Let $x^0 \in K_1$.*

(i) *If $\mathcal{L}$ is strictly pseudoconvex, then* (FJ) *conditions are sufficient for x^0 to be solution of* (P_1).

(ii) *If $\mathcal{L}$ is pseudoconvex, then* (KKT) *conditions are sufficient for x^0 to be solution of* (P_1).

(iii) *If f is pseudoconvex, every g_i, $i = 1, ..., m$, is quasiconvex (or the function $\sum_{i=1}^{m} \lambda_i g_i$ is quasiconvex) and every h_j, $j = 1, ..., p$, is both quasiconvex and quasiconcave, then* (KKT) *conditions are sufficient for x^0 to be solution of* (P_1).

Remark 2.1. Condition (iii) of Theorem 2.7 is due to Mangasarian (see [30]). Continuous functions both quasiconvex and quasiconcave (now called *quasilinear*) have been characterized in [3] by Arrow, Hurwicz and Uzawa, but without the indispensable continuity requirement (see [32]). For differentiable functions, it is not difficult to see that f is quasilinear on the open convex set $A \subset \mathbb{R}^n$ if and only if

$$x^1, x^2 \in A, \quad f(x^1) = f(x^2) \Longrightarrow \nabla f(x^1)(x^1 - x^2) = 0.$$

Remark 2.2. It is possible to require, in (iii) of Theorem 2.7, that the objective function is quasiconvex (without necessarily having a nonzero gradient at $x^0 \in K_1$). In this case, in order to obtain sufficient (KKT) conditions, we are obliged to impose more stringent conditions on the constraint functions. It is quite easy to prove the following result.

Theorem 2.8. *Let $x^0 \in K_1$ satisfy* (KKT) *conditions; if f is quasiconvex and the function $\sum_{i=1}^{m} u_i g_i + \sum_{j=1}^{p} v_j h_j$ is strictly pseudoconvex, then x^0 is solution of* (P_1).

We have to remark that Theorem 2 in [34] is not correct, as the quasiconvexity requirement of the objective function is missing. It is also easy to see that the previous (FJ) or (KKT) conditions are sufficient optimality conditions for (P_1) if the generalized convexity assumptions are given by the corresponding "pointwise" definitions at $x^0 \in K_1$, with respect to A.

Remark 2.3. From time to time some "modified" sufficient optimality conditions for (P_1) have appeared in the literature: in these conditions (both of (FJ)-type and of (KKT)-type) the related (FJ) conditions and (KKT) conditions have been modified, with the additional restriction on the nonnegativity of multipliers $v_j : v_j \geqq 0$, $j = 1, ..., p$. It is the case, for example, of the paper [36] of Singh, where the following result is proved.

Theorem 2.9. *Suppose that in* (P_1) *f is pseudoconvex, g_i, $i = 1, ...m$, and h_j, $j = 1, ..., p$, are quasiconvex, and that $x^0 \in K_1$ satisfies the following "modified"* (KKT) *conditions*

$$\nabla f(x^0) + \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{j=1}^{p} v_j \nabla h_j(x^0) = 0;$$
$$u_i g_i(x^0) = 0, \quad i = 1, ..., m;$$
$$u_i \geqq 0, \quad i = 1, ..., m; \quad v_j \geqq 0, \quad j = 1, ..., p.$$

Then x^0 solves (P_1).

In the same paper Singh presents also a Fritz John-type sufficient optimality criterion (Theorem 2.2 in [36]), which is however non correct and which has been amended by Skarpness and Sposito in [38]. Subsequently, Ben-Israel and Mond remarked in [6] that if we consider the problem

$$(\mathsf{P}_2) \qquad \begin{cases} \text{minimize } f(x) \\ \text{subject to: } g_i(x) \leqq 0, i = 1, ..., m, \\ \qquad\qquad\quad h_j(x) \leqq 0, j = 1, ..., p, \end{cases}$$

and if we denote by K_2 the feasible set of (P_2), clearly we have

$$K_1 \subset K_2 \subset K_0$$

where $K_0 = K$ (feasible set of $(\mathsf{P}) \equiv (\mathsf{P}_0)$). Let $\mathcal{C}_i(x^0)$ be a sufficient condition for a point $x^0 \in K_i$ to be an optimal solution of (P_i), $i = 0, 2$. Then it follows trivially that

$$\mathcal{C}_i(x^0), \ h(x^0) = 0, \quad i = 0, 2,$$

is a sufficient condition for x^0 to be an optimal solution also of (P_1) and there is nothing to prove if $\mathcal{C}_i(x^0)$, $i = 0, 2$, is a well-known condition.

The following result may be considered a rather general Fritz John-type sufficient optimality condition for (P_1).

Theorem 2.10. *Let $x^0 \in K_1$, f be pseudoconvex at x^0 (with respect to A), g_i, $i \in I(x^0)$, and h_j, $j = 1, ..., p$, be quasiconvex at x^0, (with respect to A). If there exist multiplers $\alpha \in \mathbb{R}$, $u_i \in \mathbb{R}$, $i = 1, ..., m$, $v_j \in \mathbb{R}$, $j = 1, ..., p$, (α, u_i, v_j) not all zero for*

$$i \in P = \{ i : i \in I(x^0), g_i \text{ is strictly pseudoconvex at } x^0 \} \text{ and}$$

$$j \in Q = \{ j : h_j \text{ is strictly pseudoconvex at } x^0 \},$$

such that (2.5) and (2.6) hold, then x^0 solves (P_1).

The proof is quite easy and is left to the reader. We point out that some results presented in [5] are not correct; see the comments by Giorgi in [18]. Finally, in the same spirit of what was remarked by Lasserre in [27] and Giorgi in [19] with respect to problem (P), it is possible to prove the following sufficient optimality conditions for (P_1), where no generalized convexity assumption is made on the constraints.

Theorem 2.11. *Let $x^0 \in K_1$, f be pseudoconvex at x^0 (with respect to A), let K_1 be convex. If there exist multipliers $u_i \geqq 0$, $i = 1, ..., m$, $v_j \in \mathbb{R}$, $j = 1, ..., p$, such that the (KKT) conditions (2.7)-(2.8) are verified, then x^0 solves (P_1).*

2.3. The invex case

Another class of generalized convex functions that has received attention in the literature is the one of *invex functions;* see, e.g., for a survey the book [33] of Mishra and Giorgi.

Definition 2.2. *A differentiable function $f : A \longrightarrow \mathbb{R}$, where $A \subset \mathbb{R}^n$ is open, is* ***invex*** *if there exists a vector function (sometimes called "scale function" or also "kernel function") $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that*

$$f(x) - f(x^0) \geqq \nabla f(x^0)\eta(x, x^0), \quad \forall x, x^0 \in A.$$

Further generalizations are possible: for example, f is *pseudoinvex* if there exists η such that

$$f(x) - f(x^0) < 0 \Longrightarrow \nabla f(x^0)\eta(x, x^0) < 0, \quad \forall x, x^0 \in A,$$

and f is *quasiinvex* if there exists η (not identically the zero element of $\mathbb{R}^n$) such that

$$f(x) - f(x^0) \leqq 0 \Longrightarrow \nabla f(x^0)\eta(x, x^0) \leqq 0, \quad \forall x, x^0 \in A.$$

Ben-Israel and Mond proved in [7] that the class of invex functions is characterized by the class of differentiable functions whose stationary points are global minimum points. Thus, for scalar functions, invexity and pseudoinvexity coincide; see also the paper [17] by Giorgi for other comments. Invex functions allow a restatement of the sufficient optimality conditions of the (KKT)-type for problem (P): if $x^0 \in K$ and (2.2)-(2.3)-(2.4) hold, then x^0 solves (P) if either:

1) f and every g_i, $i = 1, ..., m$, are invex with respect to the same η;

2) f is invex, every g_i, $i \in I(x^0)$, is quasiinvex, with respect to the same η;

3) f is invex, $\sum_{i=1}^m u_i g_i$ is quasiinvex, with respect to the same η;

4) $f + \sum_{i=1}^m u_i g_i$ is invex.

It appears from the previous results the requirement that the functions involved have to be, e.g., invex with respect to the same function η. In [31] and in [33] the class of functions invex with respect to the same scale function η has been characterized.

Theorem 2.12. *Let $f_1, ..., f_p$ be differentiable functions on the open set $A \subset \mathbb{R}^n$. The following statements are equivalent:*

(i) *The functions* $f_1, ..., f_p$ *are invex with respect to the same scale function* η.

(ii) *The functions* $\sum_{i=1}^{p} \lambda_i f_i$ $(\lambda_i \geqq 0,\ i = 1, ..., p)$ *are invex with respect to the same* η.

(iii) *The functions* $\sum_{i=1}^{p} \lambda_i f_i$ $(\lambda_i \geqq 0,\ i = 1, ..., p)$ *are invex.*

(iv) *For every* $(\lambda_1, ..., \lambda_p)$, $\lambda_i \geqq 0$, $i = 1, ..., p$, *every stationary point of* $\sum_{i=1}^{p} \lambda_i f_i$ *is a global minimum.*

A device to avoid the requirement of a common scale function η, has been proposed by Jeyakumar and Mond in [24]; we shall mention their class of functions in the next section. A recent critical paper on invex functions is the one by Zalinescu (see [39]).

2.4. Two useless generalized convex functions

Hanson and Mond have defined in [23] another class of generalized convex functions, in order to generalize the concept of invex functions. A functional $f : A \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $A \subset \mathbb{R}^n$, is called *sublinear* with respect to the second component if:

$$f(x; g + d) \leqq f(x; g) + f(x; d), \quad \forall g, d \in \mathbb{R}^n,\ \forall x \in A$$

and

$$f(x; \alpha g) = \alpha f(x; g), \quad \forall \alpha \geqq 0, \quad \forall g, d \in \mathbb{R}^n,\ \forall x \in A.$$

Definition 2.3. *Let* $f : A \longrightarrow \mathbb{R}$*,* A *open set of* $\mathbb{R}^n$*, be differentiable;* f *is called **F-convex** if there exists a sublinear functional* $F(x, u, \cdot) : A \times A \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that, for all* $x.u \in A$*, we have*

$$f(x) - f(u) \geqq F(x, u, \nabla f(u)).$$

However, this one is not a generalization of the class of invex functions, as Craven and Glover have proved in [14] the equivalence between the class of invex functions and the class of the functions considered by Hanson and Mond in [23]. Caprari extended in [9] the said equivalence result to other classes of functions ("ρ-invex functions"), both for the smooth case and for the Lipschitz nonsmooth case. The list of papers which use the F-generalized convex functions, in the convinction that these classes are more general than the corresponding classes of generalized invex functions, is too long to be reported here.

Bector and Singh introduced in [4] the class of *B-vex functions.*

Definition 2.4. *A function $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be **B-vex** with respect to $B = (b_1, b_2)$, if for any $x, u \in A$ and for all $\lambda \in [0, 1]$, we have*

$$f(\lambda x + (1-\lambda)u) \leqq b_1(x, u, \lambda) f(x) + b_2(x, u, \lambda) f(u),$$

where $b_1(x, u, \lambda) + b_2(x, u, \lambda) = 1$, $b_1(x, u, \lambda) \geqq 0$. $b_2(x, u, \lambda) \geqq 0$.

It is not difficult to prove (see [22, 28]) that this class of functions coincides with the class of quasiconvex functions.

3. Sufficient first-order optimality conditions in vector optimization

3.1. Generalized convexity for vector-valued functions

We first introduce the usual Pareto ordering in $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, we write

- $x \geqq y$, if $x_i \geqq y_i$, $i = 1, ..., n$;
- $x \geq y$, if $x \geqq y$, but $x \neq y$;
- $x > y$, if $x_i > y_i$, $i = 1, ..., n$.

If $y = 0 \in \mathbb{R}^n$, then $x \geqq 0$ is a *nonnegative vector,* $x \geq 0$ is a *semipositive vector,* $x > 0$ is a *positive vector.* The notations $x \leqq y$, $x \leq y$, $x < y$, are obvious.

Given the differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^s$, by $Jf(x)$ we denote the Jacobian matrix of f, evaluated at x.

There are many ways to introduce generalized convex functions for vector-valued functions. An elementary way is to adopt "component-wise definitions": e. g. $f : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ is quasiconvex on the open convex set $A \subset \mathbb{R}^n$, if every component f_i, $i = 1, ..., s$, is quasiconvex on A. But "vector definitions" are usually more general and interesting. We adopt the following ones (f is a differentiable vector function $f : A \longrightarrow \mathbb{R}^s$, with $A \subset \mathbb{R}^n$ open convex set).

- The function f is said to be *V-convex* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \geqq Jf(x^0)(x - x^0).$$

It is easy to prove that f is V-convex if and only if every component of f is convex, i. e. if and only if f is "component-wise convex".

- The function f is said to be *V-quasiconvex* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \leqq 0 \Longrightarrow Jf(x^0)(x - x^0) \leqq 0.$$

If every component of f is quasiconvex, then f is V-quasiconvex, but the converse is not true.

- The function f is said to be V_1*-pseudoconvex* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \leq 0 \Longrightarrow Jf(x^0)(x - x^0) \leq 0.$$

- The function f is said to be V_2*-pseudoconvex* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \leq 0 \Longrightarrow Jf(x^0)(x - x^0) < 0.$$

- The function f is said to be V_3*-pseudoconvex* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) < 0 \Longrightarrow Jf(x^0)(x - x^0) < 0.$$

When $s = 1$, the three definitions above collapse to the usual definition of Mangasarian (see [30]). If every component of f is pseudoconvex, then f is V_3-pseudoconvex and also V_1-pseudoconvex; if every component of f is strictly pseudoconvex, then f is V_2-pseudoconvex; if every component of f is quasiconvex and at least one is strictly pseudoconvex, then f is V_1-pseudoconvex. The converse of these statements are not true in general.

The following diagram shows the inclusion relationships between the classes of functions introduced above.

$$\begin{array}{ccc}
 & V\text{-}convex & \Rightarrow \quad V\text{-}quasiconvex \\
 & \Downarrow & \Downarrow \\
V_1\text{-}pseudoconvex & & V_3\text{-}pseudoconvex \\
 & \Uparrow & \Uparrow \\
 & V_2\text{-}pseudoconvex &
\end{array}$$

In general, there are not inclusion relationships between V_1-pseudoconvex functions and V_3-pseudoconvex functions, nor between V-convex functions and V_2-pseudoconvex functions.

- The function f is said to be V*-quasilinear* on A if, for all $x, x^0 \in A$:

$$f(x) - f(x^0) = 0 \Longrightarrow Jf(x^0)(x - x^0) = 0.$$

3.2. First-order sufficient optimality conditions for vector optimization problems

Given $f : A \longrightarrow \mathbb{R}^s$, f differentiable on the open set $A \subset \mathbb{R}^n$, and the unconstrained vector optimization problem

$$(\mathsf{VP})_0 \qquad\qquad V\text{-}\mathsf{minimize}\, f(x),$$

then $x^0 \in A$ is said to be an *efficient solution* or *Pareto optimal solution* or *nondominated solution* for $(\mathsf{VP})_0$ if there exists no $x \in A$ such that

$$f(x) - f(x^0) \leq 0;$$

$x^0 \in A$ is said to be a *weakly efficient solution* or *Pareto weak optimal solution* for $(\mathsf{VP})_0$ if there exists no $x \in A$ such that

$$f(x) - f(x^0) < 0.$$

It is clear that an efficient solution is also a weakly efficient solution. If the previous conditions are verified in a neighborhood of x^0, we have the corresponding "local" definitions. A well-known result concerning $(\mathsf{VP})_0$ is the following one (see [12]).

Theorem 3.1. *If $x^0 \in A$ is a local weakly efficient point for $(\mathsf{VP})_0$, then there exists $y \geq 0$ such that*

$$yJf(x^0) = 0. \tag{3.1}$$

A point $x^0 \in A$ which verifies (3.1) is called a *vector stationary point* for $(\mathsf{VP})_0$. If f is V_2-pseudoconvex and $x^0 \in A$ is a vector stationary point for $(\mathsf{VP})_0$, then x^0 is an efficient solution for $(\mathsf{VP})_0$. If f is V_3-pseudoconvex and $x^0 \in A$ is a vector stationary point for $(\mathsf{VP})_0$, then x^0 is a weakly efficient solution for $(\mathsf{VP})_0$. The same result holds if f is V-convex. If in (3.1) we have $y > 0$, then $x^0 \in A$, x^0 vector stationary point for $(\mathsf{VP})_0$, is an efficient solution for $(\mathsf{VP})_0$, if f is V_1-pseudoconvex.

By means of the vector notations adopted, we can introduce the following vector optimization problem, with both inequality and equality constraints:

$$(\mathsf{VP})_1 \qquad \begin{cases} V\text{-}\mathsf{minimize}\, f(x) \\ \text{subject to: } g(x) \leqq 0, \\ \qquad\qquad\quad h(x) = 0, \end{cases}$$

where $f : A \longrightarrow \mathbb{R}^s$, $g : A \longrightarrow \mathbb{R}^m$ and $h : A \longrightarrow \mathbb{R}^p$, A open set of $\mathbb{R}^n$, f and g are differentiable on A and h is continuously differentiable on A. We have the following well-known necessary optimality conditions (of the "Fritz John-type") for $(\mathsf{VP})_1$.

Theorem 3.2. *If the feasible point x^0 is a local weakly efficient solution for $(\mathsf{VP})_1$, then there exist multipliers $t \in \mathbb{R}^s$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$, $t \geqq 0$, $u \geqq 0$, $(t, u, v) \neq 0$, such that*

$$tJf(x^0) + uJg(x^0) + vJh(x^0) = 0, \tag{3.2}$$

$$ug(x^0) = 0. \tag{3.3}$$

Moreover, if a suitable constraint qualification is satisfied (e. g. the rows of $Jg(x^0)$ and $Jh(x^0)$ are linearly independent), then (3.2) is verified with $t \geq 0$.

We can obtain sufficient optimality conditions for $(\mathsf{VP})_1$, in terms of "component-wise" generalized convexity.

Theorem 3.3. *Assume that the feasible point* x^0 *satisfies conditions* (3.2)-(3.3) *with* $t \geq 0$, $u \geqq 0$, $v \in \mathbb{R}^p$. *Assume that every* g_i, $i \in I(x^0)$, *is quasiconvex and that every* h_j, $j = 1, ..., p$, *is both quasiconvex and quasiconcave.*

(i) *If every* f_k, $k = 1, ..., s$, *is pseudoconvex, then* x^0 *is a weakly efficient solution for* $(\mathsf{VP})_1$.

(ii) *If every* f_k, $k = 1, ..., s$, *is strictly pseudoconvex, then* x^0 *is an efficient solution for* $(\mathsf{VP})_1$.

Let us now consider the following "restricted vector Fritz John-Lagrange function":

$$V\mathcal{L}(x) = tf(x) + ug(x) + vh(x),$$

with $t \in \mathbb{R}^s$, $t \geqq 0$; $u \in \mathbb{R}^m$, $u \geqq 0$; $v \in \mathbb{R}^p$, $(t, u, v) \neq 0$. We have the following "scalarized" sufficient optimality conditions for $(\mathsf{VP})_1$.

Theorem 3.4.

(i) *If the feasible point* x^0 *satisfies conditions* (3.2)-(3.3) *with* $t \geqq 0$; $u \geqq 0$; $v \in \mathbb{R}^p$, $(t, u, v) \neq 0$, *and* $V\mathcal{L}(x)$ *is strictly pseudoconvex, then* x^0 *is an efficient solution for* $(\mathsf{VP})_1$.

(ii) *If the feasible point* x^0 *satisfies conditions* (3.2)-(3.3) *with* $t \geq 0$; $u \geqq 0$; $v \in \mathbb{R}^p$, $(t, u, v) \neq 0$, *and* $V\mathcal{L}(x)$ *is pseudoconvex, then* x^0 *is a weakly efficient solution for* $(\mathsf{VP})_1$.

The proof is easy and left to the reader.

Other first-order sufficient optimality conditions for $(\mathsf{VP})_1$ can be obtained by means of "vector generalized convexity".

Theorem 3.5. *Assume that the feasible point* x^0 *verifies* (3.2)-(3.3) *with* $t \geq 0$; $u \geqq 0$; *and* $v \in \mathbb{R}^p$. *Assume that* g *is V-quasiconvex and* h *is V-quasilinear.*

(i) *If* f *is* V_2*-pseudoconvex, then* x^0 *is an efficient solution for* $(\mathsf{VP})_1$.

(ii) *If* f *is* V_3*-pseudoconvex, then* x^0 *is a weakly efficient solution for* $(\mathsf{VP})_1$.

Assume that the feasible point x^0 *verifies* (3.2)-(3.3) *with* $t > 0$, $u \geqq 0$ *and* $v \in \mathbb{R}^p$. *Assume that* g *is V-quasiconvex and* h *is V-quasilinear.*

(iii) *If* f *is* V_1*-pseudoconvex, then* x^0 *is an efficient solution for* $(\mathsf{VP})_1$.

(iv) *If* f *is V-convex, then* x^0 *is an efficient solution for* $(\mathsf{VP})_1$.

Again the proof is easy and left to the reader (perform the proof by negation of the hypothesis and use Motzkin's theorem of the alternative).

Remark 3.1. By means of the usual device of requiring in (3.2) the non-negativity of v (and not simply that $v \in \mathbb{R}^p$), we can obtain "modified" Fritz John-type sufficient optimality conditions for $(\mathsf{VP})_1$, assuming that h is V-quasiconvex (and not V-quasilinear). For example, we have the following result.

Theorem 3.6. *Assume that the feasible point x^0 verifies* (3.2)-(3.3) *with $t > 0$, $u \geqq 0$, $v \geqq 0$. Assume that f is V_1-pseudoconvex, g and h are V-quasiconvex. Then x^0 is an efficient solution for* $(\mathsf{VP})_1$.

We point out that Majumdar (see [29]), with the praiseworth intention to correct a sufficient optimality result of Singh (see [37]), in turn presents two non correct theorems. These have been amended by Kim and others in [26] and by Giorgi, Jiménez and Novo in [21].

3.3. The vector invex case

The notion of scalar invexity (for differentiable functions) can be extended to the vector case as follows.

Definition 3.1. *Let $f : A \longrightarrow \mathbb{R}^s$ be differentiable on the open set $A \subset \mathbb{R}^n$. The function f is V-invex on A if there exists a function $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that*

$$f(x) - f(x^0) \geqq Jf(x^0)\eta(x, x^0),\ \forall x, x^0 \in A.$$

It is easy to see that V-invexity is equivalent to the invexity, *with respect to the same η*, of each component of f. The properties of V-convex and V-pseudoconvex functions related to vector stationary points, can be extended to V-invex functions.

Theorem 3.7. *If f is V-invex on A and $x^0 \in A$ is a vector stationary point, then x^0 is a weakly efficient point for* $(\mathsf{VP})_0$. *Moreover, if* (3.1) *holds with $y > 0$, then x^0 is an efficient point for* $(\mathsf{VP})_0$.

We point out that, unlike scalar invex functions (characterized by the property that every stationary point is a global minimum point), Theorem 3.7 gives only sufficient conditions. In order to characterize the class of vector functions for which a vector stationary point is a weakly efficient point for $(\mathsf{VP})_0$, Osuna-Gomez and others (see [35]) and Arana-Jimenez and others (see [1]) have introduced the notion of V-pseudoinvex functions. In order to be consistent with the notions of V-pseudoconvexity previously introduced, we accept the following definitions. Let $f : A \longrightarrow \mathbb{R}^s$ be differentiable on the open set $A \subset \mathbb{R}^n$.

- The function f is said to be V_1-*pseudoinvex* on A, if there exists a scale function $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \leq 0 \Longrightarrow Jf(x^0)\eta(x, x^0) \leq 0.$$

- The function f is said to be V_2-*pseudoinvex* on A, if there exists a scale function $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that, for all $x, x^0 \in A$:

$$f(x) - f(x^0) \leq 0 \Longrightarrow Jf(x^0)\eta(x, x^0) < 0.$$

- The function f is said to be V_3-*pseudoinvex* on A, if there exists a scale function $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that, for all $x, x^0 \in A$:

$$f(x) - f(x^0) < 0 \Longrightarrow Jf(x^0)\eta(x, x^0) < 0.$$

Unlike the scalar case, the class of V-invex functions and any class of the previously introduced functions, do not coincide. For example, it can be proved (see [1]) that the class of V-invex functions (with respect to a certain scale function η) is strictly contained in the class of V_3-pseudoinvex functions (with respect to the same η). Osuna-Gomez and others (see [35]) and Arana-Jiménez and others (see [1]) proved, respectively, point (i) and point (ii) of the following theorem.

Theorem 3.8.

(i) *A vector function* $f : A \longrightarrow \mathbb{R}^s$ *is* V_3*-pseudoinvex on* A *if and only if every vector stationary point of* $(\mathsf{VP})_0$ *is a weakly efficient point for* $(\mathsf{VP})_0$.

(ii) *A vector function* $f : A \longrightarrow \mathbb{R}^s$ *is* V_2*-pseudoinvex on* A *if and only if every vector stationary point of* $(\mathsf{VP})_0$ *is an efficient point for* $(\mathsf{VP})_0$.

We can obtain sufficient optimality conditions for $(\mathsf{VP})_1$, under the assumption that the objective function f is V_1-pseudoinvex or V_2-pseudoinvex or V-invex (and therefore V_3-psedoinvex, with respect to the same η). For example, it is quite easy to prove the following results.

Theorem 3.9.

(i) *Assume that the feasible point* x^0 *verifies* (3.2)-(3.3) *with* $t > 0$, $u \geqq 0$, $v \geqq 0$. *Assume that* f *is* V_1*-pseudoinvex, with respect to* η*, and that* g *and* h *are* V*-invex, with respect to the same* η*. Then* x^0 *is an efficient solution of* $(\mathsf{VP})_1$.

(ii) *Assume that the feasible point* x^0 *verifies* (3.2)-(3.3) *with* $t \geq 0$, $u \geqq 0$, $v \geqq 0$. *Assume that* f *is* V_2*-pseudoinvex, with respect to* η*, and that* g *and* h *are* V*-invex, with respect to the same* η*. Then* x^0 *is an efficient solution of* $(\mathsf{VP})_1$.

Remark 3.2. We have already remarked that the use of invex or generalized invex functions in optimization problems (both scalar and vector) requires the same scale function η for the objective function and the constraints. In order to avoid this difficulty, Jeyakumar and Mond introduced in [24] the concept of V-invex functions, a class of functions we call *JM-invex*, in order to make no confusion with the classes previously introduced.

Definition 3.2. *A vector-valued function* $f : A \longrightarrow \mathbb{R}^s$, A *open set of* $\mathbb{R}^n$, *is said to be* ***JM-invex on*** A *if there exist functions* $\eta : A \times A \longrightarrow \mathbb{R}^n$ *and* $\alpha_i : A \times A \longrightarrow \mathbb{R}_+ \setminus \{0\}$, *such that*

$$f_i(x) - f_i(x^0) - \alpha_i(x, x^0)\nabla f_i(x^0)\eta(x, x^0) \geqq 0, \quad \forall x, x^0 \in A, \quad i = 1, ..., s.$$

In other words, the scale function can be decomposed in a fixed part, common to all components, and a variable part, described by $\alpha_i(x, x^0)$. In the same paper the authors give several examples of JM-invex functions and obtain sufficient conditions and duality conditions for a vector optimization problem with inequality constraints. An interesting results proved by Jeyakumar and Mond (see [24, Proposition 2.2]) is the following one.

Theorem 3.10. *Let* $f : A \longrightarrow \mathbb{R}^s$ *be JM-invex. Then* $x^0 \in A$ *is a weakly efficient solution of* $(\mathsf{VP})_0$ *if and only if* x^0 *is a vector stationary point for* $(\mathsf{VP})_0$.

The class of JM-invex functions has been characterized by Craven in [13].

3.4. A nonsmooth vector invex case

Invex functions have been extended to nonsmooth functions in various ways, see, e.g., [10, 20, 25]. Here we consider the extension to locally Lipschitz functions, endowed with generalized Clarke derivatives (see [11]).

The *generalized Clarke directional derivative* of a locally Lipschitz function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at x in the direction d is denoted by $f^o(x, d)$ and is given by

$$f^o(x, d) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(y + td) - f(y)}{t}.$$

The *Clarke generalized subgradient* of f at x is denoted by $\partial f(x)$ and is given by

$$\partial f(x) = \{\xi : f^o(x, d) \geqq \xi d, \quad \forall d \in \mathbb{R}^n\}.$$

Definition 3.3. *A locally Lipschitz function $f : A \longrightarrow \mathbb{R}$, A open set of $\mathbb{R}^n$, is* ***Clarke-invex (C-invex)*** *on A if, for all x, $x^0 \in A$ there exists a function $\eta : A \times A \longrightarrow \mathbb{R}^n$ such that*

$$f(x) - f(x^0) \geqq \xi\eta(x, x^0), \quad \xi \in \partial f(x^0).$$

We now consider a vector optimization problem with inequality constraints:

$$(\mathsf{VP}) \qquad \begin{cases} V\text{-}\mathsf{minimize}\, f(x) \\ \text{subject to } g(x) \leqq 0, \end{cases}$$

where $f : A \longrightarrow \mathbb{R}^s$ and $g : A \longrightarrow \mathbb{R}^m$ are locally Lipschitz functions on the open set $A \subset \mathbb{R}^n$.

We extend to the nonsmooth case the definition given by Jeyakumar and Mond in [24] for the differentiable case.

Definition 3.4. *The problem* (VP) *is said to be* ***Clarke-Jeyakumar-Mond invex (CJM-invex)*** *if there exist $\eta : A \times A \longrightarrow \mathbb{R}^n$ and $\alpha_i, \beta_j : A \times A \longrightarrow \mathbb{R}_+ \setminus \{0\}$ such that, for every x, $x^0 \in A$:*

$$\begin{array}{rcl} f_i(x) - f_i(x^0) - \alpha_i(x, x^0)\xi_i\eta(x, x^0) & \geqq & 0, \quad \xi_i \in \partial f_i(x^0) \\ g_j(x) - g_j(x^0) - \beta_j(x, x^0)\zeta_j\eta(x, x^0) & \geqq & 0, \quad \zeta_j \in \partial g_j(x^0). \end{array}$$

Theorem 3.11. *Let the following nonsmooth Karush-Kuhn-Tucker conditions for* (VP) *be satisfied at a feasible point x^0 :*

$$0 \in \sum_{i=1}^{s} \tau_i \partial f_i(x^0) + \sum_{j=1}^{m} \lambda_j \partial g_j(x^0),$$

$$\begin{array}{l} \lambda_j g_j(x^0) = 0, \quad j = 1, ..., m, \\ \tau \geq 0, \quad \lambda \geqq 0. \end{array}$$

If (VP) *is CMJ-invex, then x^0 is a weakly efficient point for* (VP).

Proof. The condition $0 \in \sum_{i=1}^{s} \tau_i \partial f_i(x^0) + \sum_{j=1}^{m} \lambda_j \partial g_j(x^0)$ implies there exist $\xi_i \in \partial f_i(x^0)$, $\zeta_j \in \partial g_j(x^0)$ such that

$$\sum_i \tau_i \xi_i + \sum_j \lambda_j \zeta_j = 0. \tag{3.4}$$

From *CMJ*-invexity of (VP) we have

$$\begin{array}{rcl} f_i(x) - f_i(x^0) - \alpha_i(x, x^0)\xi_i\eta(x, x^0) & \geqq & 0, \quad \xi_i \in \partial f_i(x^0) \\ g_j(x) - g_j(x^0) - \beta_j(x, x^0)\zeta_j\eta(x, x^0) & \geqq & 0, \quad \zeta_j \in \partial g_j(x^0). \end{array}$$

Since $\alpha_i(x,x^0) > 0$, $\beta_j(x,x^0) > 0$, $\tau_i \geqq 0$ (not all zero) and $\lambda_j \geqq 0$, the above system of inequalities implies

$$\sum_i \frac{\pi_i(f_i(x) - f_i(x^0))}{\alpha_i(x,x^0)} + \sum_j \frac{\lambda_j(g_j(x) - g_j(x^0))}{\beta_j(x,x^0)} \geqq$$
$$\geqq (\sum_i \tau_i \xi_i + \sum_j \lambda_j \zeta_j)\eta(x,x^0)$$

for all $\sum_i \tau_i \xi_i + \sum_j \lambda_j \zeta_j \in \sum_{i=1}^s \tau_i \partial f_i(x^0) + \sum_{j=1}^m \lambda_j \partial g_j(x^0)$. Now, for ξ_i and ζ_j satisfying (3.4) we get

$$\sum_i \tau_i(f_i(x) - f_i(x^0)/\alpha_i(x,x^0) + \sum_j \lambda_j(g_j(x) - g_j(x^0))/\beta_j(x,x^0) \geqq 0. \quad (3.5)$$

Now suppose x^0 is not a weakly efficient solution for (VP). Then there exists a feasible x such that $f_i(x) < f_i(x^0)$, $i = 1, ..., s$. This implies

$$\sum_{i=1}^s \tau_i(f_i(x) - f_i(x^0)/\alpha_i(x,x^0) < 0. \quad (3.6)$$

Also since x is feasible in (VP) and $\lambda_j g_j(x^0) = 0$, $j = 1, ..., m$, we have

$$\sum_j \lambda_j(g_j(x) - g_j(x^0))/\beta_j(x,x^0) \leqq 0. \quad (3.7)$$

Adding (3.7) to (3.6) gives

$$\sum_i \tau_i(f_i(x) - f_i(x^0)/\alpha_i(x,x^0) + \sum_j \lambda_j(g_j(x) - g_j(x^0))/\beta_j(x,x^0) < 0,$$

contradicting (3.5). □

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