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# A class of parabolic evolutional inequalities and application to contact problem

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**Abstract -** We study a class of evolutional variational inequalities of parabolic type where we establish a general existence and uniqueness result. Then we apply the abstract result to solve a dynamic thermal subdifferential contact problem with friction, for time depending nonlinear long memory visco-elastic materials, with or without the clamped condition, which can be put into a general model of system defined by a second order evolution inequality, coupled with a first order evolution equation. We present and state an existence and uniqueness of weak solution, by using fixed point methods, monotonicity and convexity.

**Key words and phrases :** Parabolic evolution inequality, time depending long memory thermo-visco-elasticity, sub-differential contact condition, non clamped condition, dynamic process, fixed point.

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## 1. Introduction

Since the years 1970, fruitful mathematical studies on deformable bodies, for elastic or viscoelastic materials, within the weak formulation and variational inequalities framework, were initiated by Duvaut and Lions, followed by Kikuchi, Oden and Martins, Panagiotopoulos, Ciarlet, in the pioneering works [6, 7, 10, 11, 13]. By taking into account the parameter of the temperature field, Panagiotopoulos studied unilateral boundary value problems in linear thermo-elasticity, see [13]. Later further extensions to non convex contact conditions with non-monotone and possible multi-valued constitutive laws led to the domain of non-smooth mechanics, within the framework of the so-called hemivariational inequalities, for a mathematical treatment as well as mechanical modeling we refer to [12, 14].

This work is a continuation of the results obtained in [1]. In [1] the authors studied a class of dynamic linear viscoelastic thermal problems, without the clamped condition, where the contact is governed by a general sub-differential condition. An existence and uniqueness result of weak solution on the displacement and temperature fields has been proved, and some numerical simulations have been performed.

Here the new feature in this paper is that we extend the mechanical problem to time depending nonlinear thermo-viscoelastic law, with or without the clamped condition. We investigate a new approach to prove an existence and uniqueness result on the displacement and temperature fields, using monotonicity, convexity of the operators and fixed point methods widely used in the contact literature, see e.g. [8].

The paper is organized as follows. In Section 2 we describe a key result concerning the solvability of a class of time depending evolutional variational inequalities of parabolic type. In Section 3 we give a typical application to contact problems. We describe the mechanical problem, and derive the corresponding variational formulation. Then after specifying the assumptions on the different data we state an existence and uniqueness of weak solution. Finally in Section 4 we give the proof of the claimed result.

## 2. A parabolic differential inclusion

In the study of many contact problems, parabolic differential inclusions are frequently useful. Various abstract formulations concerning the existence and uniqueness result on parabolic variational inequalities of the second kind could be found in the literature, depending on the assumptions on the operators and data (see e.g. [3], [7], [9], [15]). Here for our purpose we need an existence and uniqueness result for a special class of time depending nonlinear evolutional inequalities of parabolic type. We give the statement of the key result and provide the main steps of the proof. A similar result could be found in [15, II/B p. 893].

Let $H$ and $V \subset H$ two Hilbert spaces, $V'$ be the dual space of $V$. We denote in the sequel by $(\cdot,\cdot)_{\mathcal{E}}$ the inner product and by $\|\cdot\|_{\mathcal{E}}$ the associated norm of any Hilbert space $\mathcal{E}$. Identifying then $H$ with its own dual, we suppose a Gelfand evolution triple (see e.g. [15, II/A p. 416]):

$$V \subset H \equiv H' \subset V'$$

where the inclusions are continuous and dense. Finally, we use the notation $\langle \cdot,\cdot \rangle_{V'\times V}$ to represent the duality pairing between $V'$ and $V$. Then we have

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V'\times V} = (\boldsymbol{u}, \boldsymbol{v})_H, \ \forall\, \boldsymbol{u} \in H, \forall\, \boldsymbol{v} \in V.$$

Denote by $C(0,T;X)$ the class of continuous functions defined on $(0,T)$; and $C^m(0,T;X)$, $m \in \mathbb{N}^*$ the set of $m$ times continuously differentiable functions defined on $(0,T)$ with values in some set $X$. Also $L^p(0,T;X)$, $W^{m,p}(0,T;X)$ represent the classical $L^p$ and Sobolev spaces defined on $(0,T)$ with values in $X$, where $m \in \mathbb{N}$, $1 \le p \le +\infty$.

We consider the following operators and data with the corresponding assumptions.

Let $T > 0$, $A : (0,T)\times V \longrightarrow V'$. Denoting by $A(t) = A(t,\cdot)$ we suppose

$$\begin{cases} \text{(i) the measurability:} \\ \quad \forall(\boldsymbol{x},\boldsymbol{y}) \in V\times V,\ t\in(0,T) \longmapsto (A(t)\boldsymbol{x},\boldsymbol{y})_{V'\times V} \ \text{ is measurable;} \\ \text{(ii) the linearly increase:} \\ \quad \exists c_1>0,\ \exists c_2 \in L^2(0,T;\mathbb{R}^+),\ \forall \boldsymbol{x}\in V, \text{ for a.e. } t\in(0,T), \\ \quad \|A(t)\,\boldsymbol{x}\|_{V'} \le c_1\,\|\boldsymbol{x}\|_V + c_2(t); \\ \text{(iii) the hemicontinuity:} \\ \quad \forall \boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z}\in V, \text{ for a.e. } t\in(0,T), \\ \quad \langle A(t)\,(\boldsymbol{x}+\tau\,\boldsymbol{y}),\boldsymbol{z}\rangle_{V'\times V} \longrightarrow \langle A(t)\,\boldsymbol{x},\boldsymbol{z}\rangle_{V'\times V},\ \text{ as } \tau\longrightarrow 0; \\ \text{(iv) the coerciveness:} \\ \quad \exists\alpha\in\mathbb{R},\ \exists\beta>0,\ \forall(\boldsymbol{x},\boldsymbol{y})\in V\times V, \text{ for a.e. } t\in(0,T), \\ \quad \langle A(t)\,\boldsymbol{x}-A(t)\,\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}\rangle_{V'\times V} + \alpha\,\|\boldsymbol{x}-\boldsymbol{y}\|_H^2 \ge \beta\,\|\boldsymbol{x}-\boldsymbol{y}\|_V^2. \end{cases} \tag{2.1}$$

Let $\psi : (0,T)\times V \longrightarrow \mathbb{R}$ satisfying

$$\begin{cases} \text{(i)}\ \ \forall \boldsymbol{w}\in V,\ t\in(0,T)\longmapsto \psi(t,\boldsymbol{w}) \text{ is Lebesgue measurable;} \\ \text{(ii)}\ \exists d>0,\ \exists c\in L^2(0,T;\mathbb{R}^+),\ \forall\boldsymbol{w}\in V, \text{ a.e. } t\in(0,T), \\ \qquad |\psi(t,\boldsymbol{w})| \le c(t) + d\,\|\boldsymbol{w}\|_V; \\ \text{(iii) for a.e. } t\in(0,T),\ \ \psi(t,\cdot) \text{ is convex on } V. \end{cases} \tag{2.2}$$

Let

$$\mathcal{F} \in L^2(0,T;V'). \tag{2.3}$$

and

$$\boldsymbol{v}_0 \in V. \tag{2.4}$$

The key theorem we will use is the following.

**Theorem 2.1.** *Consider $A$, $\psi$, $\mathcal{F}$, $\boldsymbol{v}_0$ verifying the hypotheses* (2.1), (2.2), (2.3) *and* (2.4). *Then there exists an unique solution $\boldsymbol{v}$ satisfying:*

$$\begin{cases} \boldsymbol{v}\in L^2(0,T;V)\cap W^{1,2}(0,T;V')\cap C(0,T;H); \\ \langle \dot{\boldsymbol{v}}(t),\boldsymbol{w}-\boldsymbol{v}(t)\rangle_{V'\times V} + \langle A(t)\,\boldsymbol{v}(t),\boldsymbol{w}-\boldsymbol{v}(t)\rangle_{V'\times V} + \psi(t,\boldsymbol{w}) - \psi(t,\boldsymbol{v}(t)) \\ \qquad \ge \langle \mathcal{F}(t),\boldsymbol{w}-\boldsymbol{v}(t)\rangle_{V'\times V}, \quad \forall\boldsymbol{w}\in V, \quad \text{a.e. } t\in(0,T); \\ \textit{and } \boldsymbol{v}(0) = \boldsymbol{v}_0. \end{cases} \tag{2.5}$$

**Proof.** Let introduce the following mappings defined for all $\boldsymbol{w} \in V$ and a.e. $t \in (0,T)$:

$$A_1(t)\boldsymbol{w} = e^{-\alpha t} A(t)(e^{\alpha t}\boldsymbol{w}) + \alpha \boldsymbol{w};$$
$$\mathcal{F}_1(t) = e^{-\alpha t}\mathcal{F}(t);$$
$$\psi_1(t,\boldsymbol{w}) = e^{-2\alpha t}\psi(t, e^{\alpha t}\boldsymbol{w}).$$

After some algebraic manipulations we check that for a.e. $t \in (0,T)$,

$$\forall \boldsymbol{v},\, \boldsymbol{w} \in V, \quad \langle A_1(t)\boldsymbol{v} - A_1(t)\boldsymbol{w}, \boldsymbol{v} - \boldsymbol{w}\rangle_{V'\times V} \geq \beta\, \|\boldsymbol{v} - \boldsymbol{w}\|_V^2. \tag{2.6}$$

The statement in Theorem 2.1 of an unique solution $\boldsymbol{v}$ satisfying (2.5) is equivalent to prove that there exists an unique $\boldsymbol{z}$ verifying:

$$\begin{cases} \boldsymbol{z} \in L^2(0,T;V) \cap W^{1,2}(0,T;V') \cap C(0,T;H); \\ \langle \dot{\boldsymbol{z}}(t), \boldsymbol{w} - \boldsymbol{z}(t)\rangle_{V'\times V} + \langle A_1(t)\, \boldsymbol{z}(t), \boldsymbol{w} - \boldsymbol{z}(t)\rangle_{V'\times V} + \psi_1(t,\boldsymbol{w}) - \psi_1(t,\boldsymbol{z}(t)) \\ \qquad \geq \langle \mathcal{F}_1(t), \boldsymbol{w} - \boldsymbol{z}(t)\rangle_{V'\times V}, \quad \forall \boldsymbol{w} \in V, \quad \text{a.e. } t \in (0,T); \\ \text{and } \boldsymbol{z}(0) = \boldsymbol{v}_0, \end{cases} \tag{2.7}$$

where $\boldsymbol{z}$ and $\boldsymbol{v}$ are linked by the relation

$$\text{for a.e. } \; t \in (0,T), \quad \boldsymbol{v}(t) = e^{\alpha t}\boldsymbol{z}(t).$$

(1) Existence of $\boldsymbol{z}$.

Let $X = L^2(0,T;V)$ and $D(L) = \{\boldsymbol{w} \in L^2(0,T;V) \cap W^{1,2}(0,T;V'), \boldsymbol{w}(0) = \boldsymbol{v}_0\}$ a closed convex set of $X$, it is well known that the mapping $L : D(L) \longrightarrow X'$, $\boldsymbol{w} \longmapsto \frac{d\boldsymbol{w}}{dt}$, is maximal monotone, see e.g. [15, II/B p. 855] or [9, p. 313].

Let $\Phi : X \longrightarrow \mathbb{R}$, $\boldsymbol{w} \longmapsto \displaystyle\int_0^T \psi_1(t,\boldsymbol{w}(t))dt$. From (2.2) it follows that $\Phi$ is well defined convex lsc $\not\equiv +\infty$, thus $\partial\Phi : X \longrightarrow 2^{X'}$ is maximal monotone. On the other hand, $D(\Phi) = X$, then $D(\partial\Phi) = X$.

Let $M = L + \partial\Phi$, we have $D(M) = D(L)$. As $D(L) \cap int(D(\partial\Phi)) = D(L) \neq \emptyset$, we deduce from Rockafeller's Theorem (1970) that $M : X \longrightarrow 2^{X'}$ is maximal monotone, see e.g. [15, II/B p. 888] or [2, p. 46].

Let $T : X \longrightarrow X'$ defined by $\forall \boldsymbol{w} \in X$, a.e. $t \in (0,T)$, $T(\boldsymbol{w})(t) = A_1(t)\boldsymbol{w}(t)$. From (2.1) and the definition of $A_1$, we have that $T$ is well defined, monotone, hemicontinuous and bounded. Thus $T$ is maximal monotone with $D(T) = X$. Then $D(M) \cap int(D(T)) = D(M) \neq \emptyset$, we deduce again from Rockafeller's Theorem that $S = T + M : X \longrightarrow 2^{X'}$ is maximal monotone.

Moreover as $M$ is monotone and from (2.6) $T$ is strongly monotone, then $S$ is strongly monotone and $S^{-1} : X' \longrightarrow 2^X$ is bounded.

We conclude that $S$ is surjective (see e.g. [4]). Thus there exists $\boldsymbol{z} \in D(S) = D(L)$ such that $\mathcal{F}_1 \in S(\boldsymbol{z}) = M(\boldsymbol{z}) + T(\boldsymbol{z})$. From the last statement it is well known that $\boldsymbol{z}$ satisfies (2.7), see details in [7, p. 57] .

(2) Uniqueness of $\boldsymbol{z}$.

Let $\boldsymbol{z}_1$, $\boldsymbol{z}_2$ two solutions verifying (2.7). Taking $\boldsymbol{w} = \boldsymbol{z}_2$ in the inequality for $\boldsymbol{z}_1$, and $\boldsymbol{w} = \boldsymbol{z}_1$ in the inequality for $\boldsymbol{z}_2$, then adding the two inequalities we obtain

$$\begin{aligned}&\int_0^T \langle \dot{\boldsymbol{z}}_1(t) - \dot{\boldsymbol{z}}_2(t), \boldsymbol{z}_2(t) - \boldsymbol{z}_1(t)\rangle_{V'\times V} dt \\ &\quad + \int_0^T \langle A_1(t)\, \boldsymbol{z}_1(t) - A_1(t)\, \boldsymbol{z}_2(t), \boldsymbol{z}_2(t) - \boldsymbol{z}_1(t)\rangle_{V'\times V} dt \geq 0.\end{aligned}$$

From (2.6) we deduce that

$$\frac{1}{2}\|\boldsymbol{z}_2(T) - \boldsymbol{z}_1(T)\|_H^2 + \alpha \int_0^T \|\boldsymbol{z}_2(t) - \boldsymbol{z}_1(t)\|_V^2 dt \leq 0.$$

Thus $\boldsymbol{z}_1 = \boldsymbol{z}_2$. □

## 3. Application to contact problem

In this section we study a class of thermal contact problems with sub-differential conditions, for long memory visco-elastic materials. We describe the mechanical problems, list the assumptions on the data and derive the corresponding variational formulations. Then we state an existence and uniqueness result on displacement and temperature fields, which we will prove in the next section.

The physical setting is as follows. A visco-elastic body occupies a domain $\Omega$ in $\mathbb{R}^d$ ($d = 1$, $d = 2$ or $d = 3$) with a Lipschitz boundary $\Gamma$ that is partionned into three disjoint measurable parts, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. Here we suppose that $\text{meas}(\Gamma_1) = 0$ or $\text{meas}(\Gamma_1) > 0$, which means that $\Gamma_1$ may be an empty set or reduced to a finite set of points. We assume that a volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$ and that surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$. The body may come in contact with an obstacle, the foundation, over the potential contact surface $\Gamma_3$. The model of the contact is specified by a general sub-differential boundary condition, where thermal effects may occur in the frictional contact with the basis. We are interested in the dynamic evolution of the body.

Let us recall now some classical notations, see e.g. [7] for further details. We denote by $S_d$ the space of second order symmetric tensors on $\mathbb{R}^d$, while “ $\cdot$ ” and $|\cdot|$ will represent the inner product and the Euclidean norm on

$S_d$ and $\mathbb{R}^d$. Everywhere in the sequel the indices $i$ and $j$ run from 1 to $d$, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent variable. We also use the following notation:

$$H = \Big(L^2(\Omega)\Big)^d, \qquad \mathcal{H} = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega),\ 1 \le i, j \le d\},$$

$$H_1 = \{\, \boldsymbol{u} \in H \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\,\}, \qquad \mathcal{H}_1 = \{\, \boldsymbol{\sigma} \in \mathcal{H} \mid \mathrm{Div}\, \boldsymbol{\sigma} \in H\,\}.$$

Here $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and Div $: \mathcal{H}_1 \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by :

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by :

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \, dx,$$

$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_H.$$

Recall that $\mathcal{D}(\Omega)$ denotes the set of infinitely differentiable real functions with compact support in $\Omega$; and $W^{m,p}(\Omega)$, $H^m(\Omega) := W^{m,2}(\Omega)$, $m \in \mathbb{N}$, $1 \le p \le +\infty$ for the classical real Sobolev spaces; $L^p(U; X)$ the classical $L^p$ spaces defined on $U$ with values in $X$.

To continue, the mechanical problem is then formulated as follows.

**Problem $Q$ :** Find a displacement field $\boldsymbol{u} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \longrightarrow S_d$ and a temperature field $\xi : \Omega \times (0, T) \longrightarrow \mathbb{R}_+$ such that for a.e. $t \in (0, T)$:

$$\boldsymbol{\sigma}(t) = \mathcal{A}(t)\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(t)\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\, \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds - \xi(t)\, C_e(t) \quad \text{in} \quad \Omega \tag{3.1}$$

$$\ddot{\boldsymbol{u}}(t) = \mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \quad \text{in} \quad \Omega \tag{3.2}$$

$$\boldsymbol{u}(t) = 0 \quad \text{on} \quad \Gamma_1 \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2 \tag{3.4}$$

$$\boldsymbol{u}(t) \in U, \quad \varphi(t, \boldsymbol{w}) - \varphi(t, \dot{\boldsymbol{u}}(t)) \ge -\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{w} - \dot{\boldsymbol{u}}(t)) \quad \forall \boldsymbol{w} \in U \quad \text{on} \ \Gamma_3 \tag{3.5}$$

$$\dot{\xi}(t) - \mathrm{div}(K_c(t)\, \nabla \xi(t)) = -c_{ij}(t)\, \frac{\partial\, \dot{u}_i}{\partial\, x_j}(t) + q(t) \quad \text{on} \quad \Omega \tag{3.6}$$

$$-\, k_{ij}(t)\, \frac{\partial\, \xi}{\partial\, x_j}(t)\, n_i = k_e(t)\, (\xi(t) - \theta_R(t)) \quad \text{on} \quad \Gamma_3 \tag{3.7}$$

$$\xi(t) = \theta_a(t) \quad \text{on} \quad \Gamma_1 \cup \Gamma_2 \tag{3.8}$$

$$\xi(0) = \xi_0 \quad \text{in} \quad \Omega \tag{3.9}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \Omega \tag{3.10}$$

Here, (3.1) is the Kelving Voigt's time-dependent long memory thermo-visco-elastic constitutive law of the body, $\boldsymbol{\sigma}(t)$ the stress tensor, $\mathcal{A}(t)$ is the time-dependent viscosity operator, $\mathcal{G}(t)$ for the time-dependent elastic operator, $C_e(t) := (c_{ij}(t))$ represents the thermal expansion tensor, and $\mathcal{B}$ is the so called tensor of relaxation which defines the long memory of the material, as an important particular case, when $\mathcal{B} \equiv 0$, we find again the usual visco-elasticity of short memory. In (3.2) is the dynamic equation of motion where the mass density $\varrho \equiv 1$. The equation in (3.3) is the clamped condition and in (3.4) is the traction condition. On the contact surface, the general relation (3.5) is a sub-differential boundary condition such that

$$\mathcal{D}(\Omega)^d \subset U,$$

where $U$ represents the set of contact admissible test functions and $\mathcal{D}(\Omega)^d$ is the distribution space. We denote by $\boldsymbol{\sigma\nu}$ the Cauchy stress vector on the contact boundary and $\varphi : (0,T) \times \Gamma_3 \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is a given function. Various situations may be modeled by such a condition, see e.g. the monograph [13] or the PHD thesis [5, p. 92]. The differential equation (3.6) describes the evolution of the temperature field, where $K_c(t) := (k_{ij}(t))$ represents the thermal conductivity tensor, $q(t)$ the density of volume heat sources. The associated temperature boundary condition is given by (3.7) and (3.8), where $\theta_R(t)$ is the temperature of the foundation, $k_e(t)$ is the heat exchange coefficient between the body and the obstacle, and $\theta_a(t)$ represents the ambient temperature. Finally, $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\xi_0$ represent the initial displacement, velocity and temperature, respectively.

One may remark that since $\varphi$ is assumed real-valued, then unilateral contact, defined by indicator functions taking infinite values, is excluded. So the body is in fixed contact with the foundation of the body according to a friction law. This is consistent with the linear heat conduction modeled in (3.6).

We insist that the new feature here is that we may have the absence of the usual claimed condition. However, there is coerciveness with regard to the temperature by (3.7).

To derive the variational formulation of the mechanical problems (3.1)–(3.10) we need additional notations. Thus, let consider $V$ the closed subspace of $H_1$ defined by

$$V = \{\, \boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \,\} \cap U.$$

We remark that the subspace $V$ may be different or not to the whole space $H_1$, depending on the set $U$ of admissible contact conditions.

On $V$ we consider the inner product given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{u}, \boldsymbol{v})_H \qquad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V,$$

and let $\|\cdot\|_V$ be the associated norm, i.e.

$$\|\boldsymbol{v}\|_V^2 = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}}^2 + \|\boldsymbol{v}\|_H^2 \qquad \forall\, \boldsymbol{v} \in V.$$

It follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on $V$ and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev's trace theorem, we have a constant $C_0 > 0$ depending only on $\Omega$, and $\Gamma_3$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)} \;\leq\; C_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V.$$

For functional reason, it is convenient to shift the ambient temperature to zero on $\Gamma_1 \cup \Gamma_2$. We introduce for this propose $\theta = \xi - \theta_a$, by assuming $\theta_a \in H^1(0, T; H^1(\Omega))$. Thus we have $\forall t \in [0, T]$:

$$\xi(t) = \theta_a(t) \Longrightarrow \theta(t) = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2.$$

In what follows, we use the following change of variables:

$$\xi = \theta + \theta_a, \quad \xi_0 = \theta_0 + \theta_a(0).$$

Consider then the following spaces for the temperature field:

$$E = \{\eta \in H^1(\Omega),\ \eta = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2\}; \quad F = L^2(\Omega).$$

The spaces $E$ and $F$, endowed with their respective canonical inner product, are Hilbert spaces.

Identifying then $H$ and $F$ with their own duals, we obtain two Gelfand evolution triples (see e.g. [15, II/A p. 416]):

$$V \subset H \equiv H' \subset V', \quad E \subset F \equiv F' \subset E'$$

where the inclusions are continuous and dense.

In the study of the mechanical problem (3.1)-(3.10), we assume that the viscosity operator $\mathcal{A} : (0, T) \times \Omega \times S_d \longrightarrow S_d$, $(t, \boldsymbol{x}, \boldsymbol{\tau}) \longmapsto \mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\tau})$ satisfies

$$\left\{\begin{array}{l}
\text{(i) } \mathcal{A}(\cdot, \cdot, \boldsymbol{\tau}) \text{ is measurable on } (0, T) \times \Omega,\ \forall \boldsymbol{\tau} \in S_d; \\
\text{(ii) } \mathcal{A}(t, \boldsymbol{x}, \cdot) \text{ is continuous on } S_d \text{ for a.e. } (t, \boldsymbol{x}) \in (0, T) \times \Omega; \\
\text{(iii) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\tau}_1) - \mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\tau}_2)) \cdot (\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2) \geq m_{\mathcal{A}}\, |\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2|^2, \\
\qquad\quad \forall \boldsymbol{\tau}_1,\, \boldsymbol{\tau}_2 \in S_d,\ \text{for a.e. } (t, \boldsymbol{x}) \in (0, T) \times \Omega; \\
\text{(iv) there exists } c_0^{\mathcal{A}} \in L^2((0, T) \times \Omega; \mathbb{R}^+),\ c_1^{\mathcal{A}} > 0 \text{ such that} \\
\quad |\mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\tau})| \leq c_0^{\mathcal{A}}(t, \boldsymbol{x}) + c_1^{\mathcal{A}}\, |\boldsymbol{\tau}|,\ \forall \boldsymbol{\tau} \in S_d,\ \text{for a.e. } (t, \boldsymbol{x}) \in (0, T) \times \Omega.
\end{array}\right. \tag{3.11}$$

In this paper for every $t \in (0,T)$, $\boldsymbol{\tau} \in S_d$ we denote by $\mathcal{A}(t) = \mathcal{A}(t,\cdot,\cdot)$ a functional which is defined on $\Omega \times S_d$ and $\mathcal{A}(t)\,\boldsymbol{\tau} = \mathcal{A}(t,\cdot,\boldsymbol{\tau})$ some function defined on $\Omega$.

The elasticity operator $\mathcal{G} : (0,T) \times \Omega \times S_d \longrightarrow S_d$ satisfies:

$$\left\{\begin{array}{ll} \text{(i)} & \text{there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ & |\mathcal{G}(t,\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{G}(t,\boldsymbol{x},\boldsymbol{\varepsilon}_2)| \le L_{\mathcal{G}}|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ & \forall \boldsymbol{\varepsilon}_1,\, \boldsymbol{\varepsilon}_2 \in S_d,\ a.e.\ (t,\boldsymbol{x}) \in (0,T)\times\Omega\,; \\ \text{(ii)} & (t,\boldsymbol{x}) \longmapsto \mathcal{G}(t,\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } (0,T)\times\Omega, \forall \boldsymbol{\varepsilon} \in S_d\,; \\ \text{(iii)} & \text{the mapping } \mathcal{G}(\cdot,\cdot,\mathbf{0}) \in \mathcal{H}. \end{array}\right. \tag{3.12}$$

We put again $\mathcal{G}(t)\boldsymbol{\tau} = \mathcal{G}(t,\cdot,\boldsymbol{\tau})$ some function defined on $\Omega$ for every $t \in (0,T)$, $\boldsymbol{\tau} \in S_d$.
The relaxation tensor $\mathcal{B} : (0,T)\times\Omega\times S_d \longrightarrow S_d$, $(t,\boldsymbol{x},\boldsymbol{\tau}) \longmapsto (B_{ijkh}(t,\boldsymbol{x})\,\tau_{kh})$ satisfies

$$\left\{\begin{array}{l} \text{(i) } B_{ijkh} \in L^\infty((0,T)\times\Omega); \\ \text{(ii) } \mathcal{B}(t)\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{B}(t)\boldsymbol{\tau} \\ \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in S_d,\ \text{a.e. } t \in (0,T),\ \text{a.e. in } \Omega \end{array}\right. \tag{3.13}$$

where we denote by $\mathcal{B}(t)\boldsymbol{\tau} = \mathcal{B}(t,\cdot,\boldsymbol{\tau})$ which is defined on $\Omega$ for every $t \in (0,T)$, $\boldsymbol{\tau} \in S_d$.

We suppose the body forces and surface tractions satisfy

$$\boldsymbol{f}_0 \in L^2(0,T;H), \qquad \boldsymbol{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d) \tag{3.14}$$

For the thermal tensors and the heat sources density, we suppose that

$$C_e(t) = (c_{ij}(t)), \quad c_{ij} = c_{ji} \in W^{1,\infty}(O,T;L^\infty(\Omega)), \quad q \in L^2(0,T;L^2(\Omega)) \tag{3.15}$$

The boundary thermal data satisfy

$$k_e \in L^\infty((0,T)\times\Omega;\mathbb{R}^+), \quad \theta_R \in L^2(0,T;L^2(\Gamma_3)) \tag{3.16}$$

The thermal conductivity tensor verifies the usual symmetry end ellipticity: for some $c_k > 0$ independent on time and for all $(\xi_i) \in \mathbb{R}^d$,

$$K_c(t) = (k_{ij}(t)), \quad k_{ij} = k_{ji} \in W^{1,\infty}(O,T;L^\infty(\Omega)), \quad k_{ij}\,\xi_i\xi_j \ge c_k\,\xi_i\xi_i. \tag{3.17}$$

We assume that the initial data satisfy the conditions

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in V, \quad \theta_0 \in E. \tag{3.18}$$

On the contact surface, the following frictional contact function

$$\psi(t,\boldsymbol{w}) := \int_{\Gamma_3} \varphi(t,\boldsymbol{w})\,da$$

verifies

$$\begin{cases} \text{(i)} \quad \psi\,:\,(0,T)\times V \longrightarrow \mathbb{R} \quad \text{is well defined;} \\ \text{(ii)} \quad t\in(0,T)\longmapsto \psi(t,\boldsymbol{w}) \text{ is Lebesgue measurable } \forall \boldsymbol{w}\in V; \\ \text{(iii)} \quad |\psi(t,\boldsymbol{w})| \le c(t) + d\,\|\boldsymbol{w}\|_V,\ \forall \boldsymbol{w}\in V,\ \text{a.e. } t\in(0,T); \\ \text{(iv)} \quad \psi(t,\cdot) \text{ is convex on } V \text{ a.e. } t\in(0,T), \end{cases} \tag{3.19}$$

where $d > 0$ is some constante and $c \in L^2(0,T;\mathbb{R}^+)$.

To continue, using Green's formula, we obtain the variational formulation of the mechanical problem $Q$ in abstract form as follows.
**Problem $QV$ :** Find $\boldsymbol{u} : [0,T] \to V$, $\theta : [0,T] \to E$ satisfying a.e. $t \in (0,T)$:

$$\begin{cases} \langle \ddot{\boldsymbol{u}}(t) + A(t)\,\dot{\boldsymbol{u}}(t) + B(t)\,\boldsymbol{u}(t) + C(t)\,\theta(t),\ \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} \\ \quad + (\int_0^t \mathcal{B}(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_{\mathcal{H}} + \psi(t,\boldsymbol{w}) - \psi(t,\dot{\boldsymbol{u}}(t)) \\ \qquad \ge \langle \boldsymbol{f}(t),\ \boldsymbol{w} - \dot{\boldsymbol{u}}(t)\rangle_{V'\times V} \quad \forall \boldsymbol{w}\in V; \\ \dot{\theta}(t) + K(t)\,\theta(t) = R(t)\dot{\boldsymbol{u}}(t) + Q(t) \qquad \text{in} \quad E'; \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \theta(0) = \theta_0. \end{cases}$$

Here, the operators and functions $A(t)$, $B(t)\,:\,V \longrightarrow V'$, $C(t)\,:\,E \longrightarrow V'$, $K(t)\,:\,E \longrightarrow E'$, $R(t)\,:\,V \longrightarrow E'$, $\boldsymbol{f}\,:\,[0,T] \longrightarrow V'$, and $Q : [0,T] \longrightarrow E'$ are defined by $\forall \boldsymbol{v} \in V$, $\forall \boldsymbol{w} \in V$, $\forall \tau \in E$, $\forall \eta \in E$, a.e. $t \in (0,T)$:

$$\langle A(t)\,\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} = (\mathcal{A}(t)\,\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathcal{H}}; \tag{3.20}$$

$$\langle B(t)\,\boldsymbol{v}, \boldsymbol{w}\rangle_{V'\times V} = (\mathcal{G}(t)\,\boldsymbol{\varepsilon}\boldsymbol{v}, \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathcal{H}}; \tag{3.21}$$

$$\langle C(t)\tau, \boldsymbol{w}\rangle_{V'\times V} = -(\tau\, C_e(t),\, \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathcal{H}}; \tag{3.22}$$

$$\langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} = (\boldsymbol{f}_0(t), \boldsymbol{w})_H + (\boldsymbol{f}_F(t), \boldsymbol{w})_{(L^2(\Gamma_2))^d} - (\theta_a(t) C_e, \boldsymbol{\varepsilon}\boldsymbol{w})_{\mathcal{H}}; \tag{3.23}$$

$$\begin{aligned} \langle Q(t), \eta\rangle_{E'\times E} = &\int_{\Gamma_3} k_e(t)\,(\theta_R(t) - \theta_a(t))\,\eta\,dx + \int_\Omega (q(t) - \dot{\theta}_a(t))\,\eta\,dx \\ &- \sum_{i,j=1}^d \int_\Omega k_{ij}(t)\,\frac{\partial \theta_a(t)}{\partial x_j}\frac{\partial \eta}{\partial x_i}\,dx; \end{aligned} \tag{3.24}$$

$$\langle K(t)\,\tau, \eta\rangle_{E'\times E} = \sum_{i,j=1}^d \int_\Omega k_{ij}(t)\,\frac{\partial \tau}{\partial x_j}\,\frac{\partial \eta}{\partial x_i}\,dx + \int_{\Gamma_3} k_e(t)\,\tau\cdot\eta\,da; \tag{3.25}$$

$$\langle R(t)\,\boldsymbol{v}, \eta\rangle_{E'\times E} = -\int_\Omega c_{ij}(t)\,\frac{\partial v_i}{\partial x_j}\,\eta\,dx. \tag{3.26}$$

**Theorem 3.1.** *Assume that* (3.11)–(3.19) *hold, then there exists an unique solution* $\{\boldsymbol{u}, \theta\}$ *to problem* $QV$ *with the regularity :*

$$\begin{cases} \boldsymbol{u} \in W^{1,2}(0,T;V) \cap W^{2,2}(0,T;V') \cap C^1(0,T;H) \\ \theta \in L^2(0,T;E) \cap W^{1,2}(0,T;E') \cap C(0,T;F). \end{cases} \tag{3.27}$$

## 4. Proof of Theorem 3.1

The idea is to bring the second order inequality to a first order inequality, using monotone operator, convexity and fixed point arguments, and will be carried out in several steps.

Let us introduce the velocity variable

$$\boldsymbol{v} = \dot{\boldsymbol{u}}.$$

The system in Problem $QV$ is then written for a.e. $t \in (0,T)$:

$$\begin{cases} \boldsymbol{u}(t) = \boldsymbol{u}_0 + \displaystyle\int_0^t \boldsymbol{v}(s)\, ds; \\ \langle \dot{\boldsymbol{v}}(t) + A(t)\, \boldsymbol{v}(t) + B(t)\, \boldsymbol{u}(t) + C(t)\, \theta(t),\, \boldsymbol{w} - \boldsymbol{v}(t) \rangle_{V' \times V} \\ \; + (\displaystyle\int_0^t \mathcal{B}(t-s)\, \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}(t)))_{\mathcal{H}} + \psi(t, \boldsymbol{w}) - \psi(t, \boldsymbol{v}(t)) \\ \qquad \geq \langle \boldsymbol{f}(t),\, \boldsymbol{w} - \boldsymbol{v}(t) \rangle_{V' \times V} \quad \forall \boldsymbol{w} \in V; \\ \dot{\theta}(t) + K(t)\, \theta(t) = R(t)\, \boldsymbol{v}(t) + Q(t) \quad \text{in} \quad E'; \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, \quad \theta(0) = \theta_0, \end{cases}$$

with the regularity

$$\begin{cases} \boldsymbol{v} \in \boldsymbol{v} \in L^2(0,T;V) \cap W^{1,2}(0,T;V') \cap C(0,T;H) \\ \theta \in L^2(0,T;E) \cap W^{1,2}(0,T;E') \cap C(0,T;F). \end{cases}$$

To continue, we assume in the sequel that the conditions (3.11)–(3.19) of the Theorem 3.1 are satisfied. Let define

$$\mathcal{W} := L^2(0,T;\mathcal{H}).$$

We begin by

**Lemma 4.1.** *For all* $\eta \in \mathcal{W}$*, there exists an unique*

$$\boldsymbol{v}_\eta \in L^2(0,T;V) \cap W^{1,2}(0,T;V') \cap C(0,T;H)$$

*satisfying*

$$\begin{cases} \langle \dot{\boldsymbol{v}}_\eta(t) + A(t)\, \boldsymbol{v}_\eta(t),\, \boldsymbol{w} - \boldsymbol{v}_\eta(t) \rangle_{V' \times V} + (\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{w}) - \boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)))_{\mathcal{H}} \\ \quad + \psi(t, \boldsymbol{w}) - \psi(t, \boldsymbol{v}_\eta(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{v}_\eta(t) \rangle_{V' \times V}, \\ \qquad\qquad \forall\, \boldsymbol{w} \in V, \quad \text{a.e. } t \in (0,T); \\ \boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \end{cases} \tag{4.1}$$

*Moreover, $\exists c > 0$ such that $\forall \eta_1,\, \eta_2 \in \mathcal{W}$:*

$$\|\boldsymbol{v}_{\eta_2}(t)-\boldsymbol{v}_{\eta_1}(t)\|_H^2+\int_0^t \|\boldsymbol{v}_{\eta_1}-\boldsymbol{v}_{\eta_2}\|_V^2 \leq c\int_0^t \|\eta_1-\eta_2\|_{\mathcal{H}}^2, \quad \forall t \in [0,T]. \tag{4.2}$$

**Proof.** Let $\eta \in \mathcal{W}$. The existence and uniqueness of $\boldsymbol{v}_\eta$ follows straightly from Theorem 2.1, where we apply $\mathcal{F}$ defined by for all $t \in [0,T]$,

$$\langle \mathcal{F}(t), \boldsymbol{w}\rangle_{V'\times V} := \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} - (\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}}, \quad \forall \boldsymbol{w} \in V.$$

The assumptions in (3.14) imply that $\mathcal{F} \in L^2(0,T;V')$.

Now let $\eta_1, \eta_2 \in \mathcal{W}$. In (4.1) we take $(\eta = \eta_1,\, \boldsymbol{w} = \boldsymbol{v}_{\eta_2}(t))$, then $(\eta = \eta_2,\, \boldsymbol{w} = \boldsymbol{v}_{\eta_1}(t))$. Adding the two inequalities, we deduce that for a.e. $t \in (0;T)$:

$$\begin{aligned}
&\langle \dot{\boldsymbol{v}}_{\eta_2}(t) - \dot{\boldsymbol{v}}_{\eta_1}(t), \boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\rangle_{V'\times V} \\
&\quad + \langle A(t)\,\boldsymbol{v}_{\eta_2}(t) - A(t)\,\boldsymbol{v}_{\eta_1}(t), \boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\rangle_{V'\times V} \\
&\quad \leq -(\eta_2(t) - \eta_1(t), \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta_2}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta_1}(t)))_{\mathcal{H}}.
\end{aligned}$$

Then integrating over $(0,t)$, from (3.11)(iii) and from the initial condition on the velocity, we obtain:

$$\begin{aligned}
\forall t \in [0,T], \quad &\|\boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\|_H^2 + m_A\int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_V^2\, ds \\
&\leq -\int_0^t (\eta_2(s) - \eta_1(s), \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta_2}(s)) - \boldsymbol{\varepsilon}(\boldsymbol{v}_{\eta_1}(s)))_{\mathcal{H}}\, ds \\
&\quad + m_A\int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_H^2\, ds.
\end{aligned}$$

We conclude that $\exists c > 0$ such that $\forall \eta_1,\, \eta_2 \in \mathcal{W}$, $\forall t \in [0,T]$:

$$\begin{aligned}
&\|\boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\|_H^2 + \int_0^t \|\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)\|_V^2 ds \\
&\leq c\int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 ds + c\int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_H^2\, ds.
\end{aligned} \tag{4.3}$$

Now let fix $\tau \in [0,T]$. We have $\forall t \in [0,\tau]$:

$$\|\boldsymbol{v}_{\eta_2}(t) - \boldsymbol{v}_{\eta_1}(t)\|_H^2 \leq c\int_0^\tau \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 + c\int_0^t \|\boldsymbol{v}_{\eta_2}(s) - \boldsymbol{v}_{\eta_1}(s)\|_H^2\, ds.$$

Using then Gronwall's inequality, we obtain $\forall \tau \in [0,T]$:

$$\|\boldsymbol{v}_{\eta_2}(\tau) - \boldsymbol{v}_{\eta_1}(\tau)\|_H^2 \leq \Big(c\int_0^\tau \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2\Big)\, e^{cT}.$$

Finally, integrating the last inequality and reporting the result in (4.3), we get (4.2). $\square$

Here and below, we denote by $c > 0$ a generic constant, which value may change from lines to lines.

**Lemma 4.2.** *For all $\eta \in \mathcal{W}$, there exists an unique*

$$\theta_\eta \in L^2(0,T;E) \cap W^{1,2}(0,T;E') \cap C(0,T;F)$$

*satisfying*

$$\begin{cases} \dot\theta_\eta(t) + K(t)\,\theta_\eta(t) = R(t)\,\boldsymbol{v}_\eta(t) + Q(t), \quad \text{in} \quad E', \quad \text{a.e. } t \in (0,T), \\ \theta_\eta(0) = \theta_0. \end{cases} \tag{4.4}$$

*Moreover, $\exists c > 0$ such that $\forall \eta_1,\, \eta_2 \in \mathcal{W}$:*

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 \le c \int_0^t \|\boldsymbol{v}_{\eta_1} - \boldsymbol{v}_{\eta_2}\|_V^2, \quad \forall t \in [0,T]. \tag{4.5}$$

**Proof.** The existence and uniqueness result verifying (4.4) can be seen as a particular case of Theorem 2.1. Indeed we verify that the operator $K(t) : E \longrightarrow E'$ is linear continuous and strongly monotone, and from the expression of the operator $R(t)$,

$$\boldsymbol{v}_\eta \in L^2(0,T;V) \Longrightarrow R\,\boldsymbol{v}_\eta \in L^2(0,T;E'),$$

as $Q \in L^2(0,T;E')$ then $R\,\boldsymbol{v}_\eta + Q \in L^2(0,T;E')$.

Now for $\eta_1,\, \eta_2 \in \mathcal{W}$, we have for a.e. $t \in (0;T)$:

$$\begin{aligned} &\langle \dot\theta_{\eta_1}(t) - \dot\theta_{\eta_2}(t), \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E} \\ &\quad + \langle K(t)\,\theta_{\eta_1}(t) - K(t)\,\theta_{\eta_2}(t),\, \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E} \\ &\quad = \langle R(t)\,\boldsymbol{v}_{\eta_1}(t) - R(t)\,\boldsymbol{v}_{\eta_2}(t),\, \theta_{\eta_1}(t) - \theta_{\eta_2}(t)\rangle_{E'\times E}. \end{aligned}$$

Then integrating the last property over $(0,t)$, using the strong monotonicity of $K(t)$ and the Lipschitz continuity of $R(t) : V \longrightarrow E'$, we deduce the relation (4.5). □

**Proof of Theorem 3.1**. We have now all the ingredients to prove the Theorem 3.1. Consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ defined by for all $\eta \in \mathcal{W}$:

$$\Lambda\,\eta\,(t) = \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t))) + \int_0^t B(t-s)\,\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s))\,ds - \theta_\eta(t)\,C_e(t), \quad \forall t \in [0,T],$$

where

$$\boldsymbol{u}_\eta(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_\eta(s)\,ds,$$

$$\forall t \in [0,T]; \quad \boldsymbol{u}_\eta \in W^{1,2}(0,T;V) \cap W^{2,2}(0,T;V') \cap C^1(0,T;H).$$

Then from (3.12), (3.13), and Lemma 4.2, we deduce that for all $\eta_1,\, \eta_2 \in \mathcal{W}$, for all $t \in [0,T]$:

$$\begin{aligned} \|\Lambda\,\eta_1\,(t) - \Lambda\,\eta_2\,(t)\|_{\mathcal{H}}^2 &\le c\,\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|_F^2 + c\int_0^t \|\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)\|_V^2\,ds \\ &\le c\int_0^t \|\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)\|_V^2\,ds. \end{aligned} \tag{4.6}$$

Now using (4.6), after some algebraic manipulations, we have for any $\beta > 0$:

$$\int_0^T e^{-\beta\tau} \|\Lambda \eta_1(\tau) - \Lambda \eta_2(\tau)\|^2_{\mathcal{H}} d\tau \leq \frac{c}{\beta} \int_0^T e^{-\beta\tau} \|\eta_1(\tau) - \eta_2(\tau)\|^2_{\mathcal{H}} d\tau.$$

We conclude from the last inequality by contracting principle that the operator $\Lambda$ has a unique fixed point $\eta^* \in \mathcal{W}$. We verify then that the functions

$$\boldsymbol{u}(t) := \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_{\eta^*}, \ \forall t \in [0,T], \quad \theta := \theta_{\eta^*}$$

are solutions to problem $QV$ with the regularity (3.27), the uniqueness follows from the uniqueness in Lemma 4.1 and Lemma 4.2. □

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