Multiple solutions to asymptotically linear hamiltonian systems in $\mathbb{R}^{2N}$: a shooting-type approach

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Dedicated, with admiration and gratefulness, to Professor Jean Mawhin on the occasion of his 70th birthday

Abstract - We prove the existence of multiple solutions to the Dirichlet problem associated to an asymptotically linear hamiltonian system in $\mathbb{R}^{2N}, N \geq 1$. Solutions are distinguished by means of the Maslov index of suitable auxiliary linear systems.

Key words and phrases: multiplicity, Maslov index, phase angles.

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1. Introduction

In this paper we give a multiplicity result (Theorem 2.1) for a boundary value problem of the form

\[
\begin{cases}
Jz' = S(t, z)z, \quad z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad t \in [0, \pi] \\
x(0) = 0 = x(\pi),
\end{cases}
\]

(1.1)

where $S$ is a continuous function defined in $[0, \pi] \times \mathbb{R}^{2N}$ which takes values in the space of $2N \times 2N$ real symmetric matrices. We are concerned with the case when the nonlinearity $S$ is asymptotically linear (cf. $(H1)$ and (2.2)).

The literature for the above described problem is not very rich. In the framework of variational methods, we refer to the important papers by Benci-Fortunato [2] and Fortunato [6] (for the periodic BVP and for the Dirichlet BVP associated to second order problems, respectively). A continuation method has been used by Manásevich-Mawhin [8] for proving the existence of boundary value problems associated to strongly nonlinear systems of second order equations. In the framework of bifurcation theory, a multiplicity result has been given by the authors in [4]; as for systems of second order problems we refer to [5], which can be considered as the starting point of the present research. All the above quoted results deal with a Morse/Maslov-type index, for whom we refer to [1, 3] and references therein.
For the proof of our main result, we combine the study of the Maslov index associated to some linear systems arising from \((H1)\) and \((2.2)\) with the classical shooting method. More precisely, under the technical assumption \((H2)\), we are led to study an auxiliary linear system in \(\mathbb{R}^{2N}\) (cf. \((2.9)\)) which uncouples in \(N\) planar systems (cf. \((3.8)\)); to each of these planar systems we can then apply a version of Sturm comparison theorem and complete the proof.

We end this introductory section with a list of notations. For every integer \(k \in \{1, \ldots, N\}\), we define

\[ \Pi_k = \{ w \in \mathbb{R}^N \mid w_k = 0 \} . \]

For every vector \(w \in \mathbb{R}^N\) and for every \(j \in \{1, \ldots, N\}\), let \(\text{ORD}_j(w)\) be the \(j\)-th component of the vector obtained from \(w\) arranging its components from the smallest to the largest.

We also denote by \(J\) the usual standard \(2N \times 2N\) symplectic matrix and by \(J_2\) the \(2 \times 2\) symplectic matrix (i.e. \(J_2\) coincides with \(J\) when \(N = 1\)).

Finally, by \(M_{2N}^s\) we denote the set of \(2N \times 2N\) real symmetric matrices \(B\) such that, writing \(B\) in the form

\[ B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \]

where \(B_{ij}\) is an \(N \times N\) matrix \((i, j = 1, \ldots, N)\), the matrix \(B_{22}\) is positive definite.

### 2. Statement of the result

In this Section we consider a boundary value problem of the form

\[
\begin{cases}
Jz' = S(t, z)z, & z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \ t \in [0, \pi] \\
x(0) = 0 = x(\pi),
\end{cases}
\tag{2.1}
\]

where \(S : [0, \pi] \times \mathbb{R}^{2N} \longrightarrow M_{2N}^s\) is a continuous function such that uniqueness and global continuability of solutions to initial value problems associated to the equation in \((2.1)\) are guaranteed. We denote by \(s_{i,j}(t, z), \ (t, z) \in [0, \pi] \times \mathbb{R}^{2N}, \ i, j = 1, \ldots, 2N, \) the coefficients of the matrix \(S(t, z)\).

We assume the following hypotheses:

(H1) There exists a continuous map \(S_\infty : [0, \pi] \longrightarrow M_{2N}^s\) such that

\[ \lim_{|z| \to +\infty} S(t, z) = S_\infty(t), \ \text{uniformly in} \ t \in [0, \pi]. \]

(H2) For every \(k \in \{1, \ldots, N\}\), the matrix \(S_{[0, \pi] \times \Pi_k \times \Pi_k}\) is diagonal.
Moreover, let
\[ S_0(t) = S(t, 0), \quad \forall \ t \in [0, \pi]. \]  
(2.2)

In order to state our result, we need some preliminary facts about problem (2.1) and some related linear systems.

Indeed, let us first observe that \((H1)\) implies that the function \(S\) is bounded; as a consequence, we have the following result (cf. [5, Proposition 4.6]):

**Proposition 2.1.** For every \(R > 0\) there exists \(R' > 0\) such that for every solution \(z\) of
\[ Jz' = S(t, z)z \]
we have
\[ |z(0)| \leq R \quad \Rightarrow \quad |z(t)| \leq R', \quad \forall \ t \in [0, \pi]. \]  
(2.3)

Now, let us recall some facts about oscillatory theory for linear hamiltonian systems. Let us consider a continuous map \(B : [0, \pi] \to M_{2N}^s\) and the linear system
\[ Jz' = B(t)z, \quad z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \]  
(2.4)

We are interested in solutions of (2.4) such that
\[ x(0) = 0 = x(\pi). \]  
(2.5)

From [3, 5, 7] we know that there exist \(N\) continuous functions \(\theta_1, \ldots, \theta_N : [0, \pi] \to \mathbb{R}\) such that
\[ \theta_i(0) = 0, \quad \forall \ i = 1, \ldots, N \]
\[ \theta_1(t) \leq \ldots \leq \theta_N(t), \quad \forall \ t \in [0, \pi], \]
and (2.4)-(2.5) has a nontrivial solution if and only if there exist an integer \(j \in \{1, \ldots, N\}\) and \(h_j \in \mathbb{N}\) such that
\[ \theta_j(\pi) = h_j \pi. \]  
(2.6)

Moreover, the number of linearly independent solutions of (2.4)-(2.5) is exactly the number of indeces \(j \in \{1, \ldots, N\}\) such that \(\theta_j(\pi)/\pi \in \mathbb{N}\).

**Remark 2.1.** According to the previous results, every solution of (2.4) such that \(x(0) = 0\) satisfies also \(x(\pi) = 0\) if and only if
\[ \frac{\theta_j(\pi)}{\pi} \in \mathbb{N}, \quad \forall \ j = 1, \ldots, N. \]
Remark 2.2. When $N = 1$ the function $\theta_1$ is the usual angular coordinate associated to the planar system (2.4). In this situation $\theta_1$ satisfies a monotonicity property; indeed, let $B_1, B_2 : [0, \pi] \to M_2^{2N}$ be continuous functions and let $\theta_{1,B_1}$ and $\theta_{1,B_2}$ be the angular coordinates associated to $J_2 z' = B_1(t)z$ and $J_2 z' = B_2(t)z$, respectively.

We then have (cfr. [9, Theorem 16.1])

$$B_1(t) \leq B_2(t), \quad \forall t \in [0, \pi] \quad \Rightarrow \quad \theta_{1,B_1}(\pi) \leq \theta_{1,B_2}(\pi),$$

where the inequality $B_1(t) \leq B_2(t)$, for $t \in [0, \pi]$, means that the matrix $B_1(t) - B_2(t)$ is semi-definite negative.

Finally, for every $j \in \{1, \ldots, N\}$, let us set

$$\theta_j(\pi) = k_j \pi + \alpha_j,$$

with $k_j \in \mathbb{N}$ and $0 < \alpha_j \leq \pi$. Then, the Maslov index associated to (2.4) is the number

$$m(B) = k_1 + \ldots + k_N.$$

Now, for every $\alpha \in \mathbb{R}^N$ we denote by $z_\alpha$ the unique solution of the Cauchy problem

$$\begin{cases}
Jz' = S(t, z)z, \quad t \in (0, \pi), \quad z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \\
x(0) = 0, \quad y(0) = \alpha,
\end{cases}
\tag{2.7}$$

and let

$$S_\alpha(t) = S(t, z_\alpha(t)), \quad \forall t \in [0, \pi]. \tag{2.8}$$

Consider the linearized system

$$Jz' = S_\alpha(t)z \tag{2.9}$$

and denote by $\theta_{1,\alpha}, \ldots, \theta_{N,\alpha}$ its phase-angles. The following result is a straightforward variant of the corresponding Propositions in [5]:

Proposition 2.2. Under the previous assumptions we have:

$$\theta_{j,\alpha} \to \theta_{j,\infty}, \quad |\alpha| \to +\infty, \quad \text{in} \quad L^\infty([0, \pi]), \quad j = 1, \ldots, N,$$

$$t_{j,\alpha} \to t_{j,0}, \quad |\alpha| \to 0, \quad \text{in} \quad L^\infty([0, \pi]), \quad j = 1, \ldots, N.$$

Now, let us denote by $m_0$ and $m_\infty$ the numbers $m(S_0)$ and $m(S_\infty)$, respectively, and assume that the set

$$S = \{(h_1, \ldots, h_N) \in \mathbb{N}^N \mid m_0 + N < h_1 + \ldots + h_N < m_\infty, \quad h_1 \leq \ldots \leq h_N\}$$
is not empty. A completely analogous result holds true if the set
\( S' = \{(h_1, \ldots, h_N) \in \mathbb{N}^N \mid m_\infty + N < h_1 + \ldots + h_N < m_0, \ h_1 \leq \ldots \leq h_N \} \)
is not empty. Then, there exist \( \epsilon > 0 \) such that
\[
\epsilon < (h_1 + \ldots + h_N)\pi - (m_0 + N)\pi, \quad \epsilon < m_\infty\pi - (h_1 + \ldots + h_N)\pi,
\]
for every \( (h_1, \ldots, h_N) \in S \). From Proposition 2.2 we deduce that there exists \( \alpha_\infty > 0 \) such that
\[
|\alpha| \geq \alpha_\infty \quad \Rightarrow \quad \theta_{1,2}(\pi) + \ldots + \theta_{N,2}(\pi) > \theta_{1,\infty}(\pi) + \ldots + \theta_{N,\infty}(\pi) - \epsilon.
\]
With \( \alpha_\infty \) as above, let \( \alpha'_\infty > 0 \) as in Proposition 2.1 and define
\[
\tilde{\Pi}_k = \{ w \in \Pi_k \mid |w| \leq \alpha'\infty \}, \quad \forall \ k = 1, \ldots, N.
\]
Now, for every \( k = 1, \ldots, N \), consider the diagonal matrix
\[
S_{[0,\pi] \times \tilde{\Pi}_k \times \tilde{\Pi}_k}
\]
(recall assumption \((H2)\)) and denote by \( s^k_{i,i} \) its non zero elements, \( i = 1, \ldots, 2N \); define then
\[
\lambda^k_i = \max_{(t,z) \in [0,\pi] \times \tilde{\Pi}_k \times \tilde{\Pi}_k} s^k_{i,i}(t, z), \quad i = 1, \ldots, 2N,
\]
and the matrix
\[
\Lambda^k_i = \begin{pmatrix} \lambda^k_i & 0 \\ 0 & \lambda^k_{i+N} \end{pmatrix}, \quad i = 1, \ldots, N.
\]
Let \( \theta^k_i \) be the angular coordinate of the planar system
\[
J_2 w' = \Lambda^k_i w, \quad w \in \mathbb{R}^2, \quad i = 1, \ldots, N,
\]
and let
\[
\mu^k_i = \frac{\theta^k_i(\pi)}{\pi}, \quad i, k = 1, \ldots, N,
\]
and
\[
n_k = ORD_k(\mu^k), \quad k = 1, \ldots, N.
\]
We are now in position to state our result:

**Theorem 2.1.** Assume conditions \((H1)\) and \((H2)\). Moreover, let \( m_0, m_\infty, n_1, \ldots, n_N \) as above.
Let us define

\[ T = \left\{ (h_1, \ldots, h_N) \in \mathbb{N}^N \mid \begin{array}{c}
m_0 + N < h_1 + \ldots + h_N < m_{\infty}, \\
h_1 > n_1, \ldots, h_N > n_N, \\
h_1 \leq \ldots \leq h_N \end{array} \right\}. \]

Then, for every \( h = (h_1, \ldots, h_N) \in T \) there exist \( 2^N \) nontrivial solutions \( z_h \) of (2.1) such that the linear system

\[ Jz' = S(t, z_h(t))z \]

has Maslov index \( h_1 + \ldots + h_N \).

3. Proof and remarks

In order to prove Theorem 2.1 we use a shooting argument; to this aim, let us fix \( h = (h_1, \ldots, h_N) \in T \) and let \( \alpha \in \mathbb{R}^N \). Let \( z_\alpha \) be, as in Section 2, the solution of (2.7) and let \( S_\alpha \) and \( \theta_1, \alpha, \ldots, \theta_N, \alpha \) be as in the previous Section.

According to Remark 2.1, if \( \alpha \) is such that

\[ \theta_j(\pi) - h_j\pi = 0, \quad \forall \ j = 1, \ldots, N, \tag{3.1} \]

then every solution of (2.9) satisfies also \( x(0) = 0 = x(\pi) \); this is also true for \( z_\alpha \) (which, trivially, is a solution of (2.9)). Hence \( z_\alpha \) satisfies the boundary value problem (2.1); moreover, by the definition of the Maslov index,

\[ m(S_\alpha) = h_1 + \ldots + h_N. \]

Therefore, in order to prove the result, it is sufficient to find \( \alpha \in \mathbb{R}^N \) such that (3.1) holds true. This will be a consequence of the application of the following result on the existence of a zero of a vector field:

**Theorem 3.1.** (cfr. [5, Theorem 2.1]) Let \( 0 < r < R \) and consider the conical shell

\[ W^R_r = \{ w \in \mathbb{R}^N \mid r \leq |w| \leq R, \ x_i \geq 0, \ \forall \ i = 1, \ldots, N \}. \]

Assume that \( f : W^R_r \to \mathbb{R}^N \) is a continuous vector field. Let us suppose that the vector field \( f \) satisfies the following conditions:

\[ \begin{cases} f_1(\alpha) + \ldots + f_N(\alpha) < 0, & \text{if } \alpha \in W^R_r, \ |\alpha| = r, \\ f_1(\alpha) + \ldots + f_N(\alpha) > 0, & \text{if } \alpha \in W^R_r, \ |\alpha| = R. \end{cases} \tag{3.2} \]
Moreover, let us assume that we have

\[ f_k(\alpha) < 0, \quad \text{if} \quad \alpha \in W^R_k \cap \Pi_k, \quad (3.3) \]

for every \( k \in \{1, \ldots, N\} \). Then, there is at least a point \( \hat{\alpha} \) in the interior of \( W^R_k \) such that

\[ f(\hat{\alpha}) = 0. \]

**Remark 3.1.** According to [5], an analogous result holds true if \( W^R_k \) is replaced by the set

\[ \tilde{W}^R_r = \{ w \in \mathbb{R}^N \mid r \leq |w| \leq R \} \cap O_N, \quad (3.4) \]

where \( O_N \) is any octant of \( \mathbb{R}^N \).

Now, we will apply Theorem 3.1 to the vector field \( f \) whose components are defined by

\[ f_i(\alpha) = \theta_{i, \alpha}(\pi) - h_i \pi, \quad (3.5) \]

for every \( \alpha \in \mathbb{R}^N \) and \( i \in \{1, \ldots, N\} \). For \( \epsilon > 0 \) as in (2.10), from Proposition 2.2 we deduce that there exists \( \alpha_0 > 0, \alpha_0 < \alpha_\infty \) such that

\[ |\alpha| \leq \alpha_0 \quad \Rightarrow \quad \theta_{1, \alpha}(\pi) + \ldots + \theta_{N, \alpha}(\pi) < \theta_{1,0}(\pi) + \ldots + \theta_{N,0}(\pi) + \epsilon. \quad (3.6) \]

Let us consider the conical shell \( W_{\alpha_0}^{\alpha_\infty} \); if \( \alpha \in W_{\alpha_0}^{\alpha_\infty} \) and \( |\alpha| = \alpha_0 \), then the definition of the Maslov index \( m_0 = m(S_0) \) and (3.6) imply that

\[ \theta_{1, \alpha}(\pi) + \ldots + \theta_{N, \alpha}(\pi) < (m_0 + N)\pi + \epsilon, \]

i.e.

\[ f_1(\alpha) + \ldots f_N(\alpha) = \theta_{1, \alpha}(\pi) + \ldots + \theta_{N, \alpha}(\pi) - (h_1 + \ldots + h_N)\pi < \]

\[ < (m_0 + N)\pi - (h_1 + \ldots + h_N)\pi + \epsilon < 0, \]

by (2.10). This proves the validity of the first inequality in (3.2).

Similarly, when \( \alpha \in W_{\alpha_0}^{\alpha_\infty} \) and \( |\alpha| = \alpha_\infty \), from (2.10) and (2.11) we infer that

\[ f_1(\alpha) + \ldots f_N(\alpha) = \theta_{1, \alpha}(\pi) + \ldots + \theta_{N, \alpha}(\pi) - (h_1 + \ldots + h_N)\pi > \]

\[ > m_\infty \pi - \epsilon - (h_1 + \ldots + h_N)\pi > 0; \]

as a consequence, also the second inequality in (3.2) is satisfied.

We now check the validity of (3.3); we consider the case \( k = 1 \) (the other cases can be proved in a very similar way). Hence, let \( \alpha \in W_{\alpha_0}^{\alpha_\infty} \cap \Pi_1 \); from an easy computation it is possible to show that

\[ \alpha \in \Pi_1 \quad \Rightarrow \quad z_\alpha \in \Pi_1 \times \Pi_1. \]
Moreover, from Proposition 2.1 we also know that
\[ |\alpha| \leq \alpha_\infty \implies |z_\alpha(t)| \leq \alpha'_\infty, \quad \forall \ t \in [0, \pi]; \]
hence, we have
\[ |\alpha| \leq \alpha_\infty, \quad \alpha \in \Pi_1 \implies z_\alpha(t) \in \bar{\Pi}_1 \times \bar{\Pi}_1, \quad \forall \ t \in [0, \pi]. \]
This implies that the matrix
\[ S_\alpha(t) = S(t, z_\alpha(t)) \]
is diagonal and its entries \( s_{1,i,i} \) satisfy
\[ s_{1,i,i}(t) \leq \lambda^1_i, \quad \forall \ t \in [0, \pi], \quad \forall \ i = 1, \ldots, N. \tag{3.7} \]
Moreover, it is possible to see that the linearized system (2.9) uncouples into \( N \) planar systems
\[ J_2 w' = \tilde{S}_{\alpha,i}(t)w, \quad w \in \mathbb{R}^2, \quad i = 1, \ldots, N, \tag{3.8} \]
where \( \tilde{S}_{\alpha,i} \) is the matrix
\[ \tilde{S}_{\alpha,i} = \begin{pmatrix} s_{1,i,i} & 0 \\ 0 & s_{1,i+N,i+N} \end{pmatrix}, \quad i = 1, \ldots, N. \]
As a consequence, up to a reordering, the phase-angles \( \theta_{1,\alpha}, \ldots, \theta_{N,\alpha} \) coincide with the phase-angles \( \tilde{\theta}_{1,\alpha}, \ldots, \tilde{\theta}_{N,\alpha} \) of the planar systems (3.8). Recalling (3.7) and Remark 2.2, we infer that
\[ \tilde{\theta}_{i,\alpha}(\pi) \leq \theta^1_i(\pi), \quad \forall \ i = 1, \ldots, N; \]
hence
\[ \frac{\tilde{\theta}_{i,\alpha}(\pi)}{\pi} \leq \frac{\theta^1_i(\pi)}{\pi} = \mu^1_i, \quad \forall \ i = 1, \ldots, N, \]
by (2.12). In particular, from (2.13) we deduce that there exists \( i^* \in \{1, \ldots, N\} \) such that
\[ \mu^1_{i^*} = n_1; \]
as a consequence, we obtain
\[ \theta_{1,\alpha}(\pi) \leq \tilde{\theta}_{i^*,\alpha}(\pi) \leq n_1\pi. \]
This implies that
\[ f_1(\alpha) = \theta_{1,\alpha}(\pi) - h_1\pi \leq (n_1 - h_1)\pi < 0, \]
proving (3.3) when \( k = 1 \).

An application of Theorem 3.1 ensures the existence of \( \alpha \in W^\alpha_{\alpha_0} \) such that \( f(\alpha) = 0 \); the corresponding function \( z_\alpha \) is a solution of (2.1).

To obtain the other \( 2^N - 1 \) solutions of (2.1) it is sufficient to apply Theorem 3.1 to any set of the form (3.4).
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