On the Neumann problem involving the Hardy - Sobolev potentials

JAN CHABROWSKI

Communicated by George Dinca

Abstract - We establish the existence of solutions for the Neumann problem involving two Hardy - Sobolev potentials with singularities at two distinct points.

Key words and phrases : the Neumann problem, the Hardy - Sobolev inequality.


1. Introduction

In this paper we investigate the nonlinear Neumann problem

\[
\begin{cases}
-\Delta u = \frac{P_1(x)}{|x|^2} |u|^{2^*(t_1)-2} u + \frac{P_2(x)}{|x-\xi|^2} |u|^{2^*(t_2)-2} u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
u > 0 & \text{on } \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with a smooth boundary \( \partial \Omega \). It is assumed that \( 0, \xi \in \partial \Omega \). \( 2^*(t_j) \) denote Hardy - Sobolev exponents given by \( 2^*(t_j) = \frac{2(N-t_j)}{N-2} \), \( 0 \leq t_j \leq 2 \). In this paper we only consider the case \( 0 < t_j < 2 \). If \( t_j = 0 \) for \( j = 1, 2 \), then \( 2^*(t_j) = 2^* = \frac{2N}{N-2} \) and this problem has an extensive literature. We refer to papers [1], [2], [6], [7], [10], [26]. The existence results in the case \( t_1 = 0 \) and \( 0 < t_2 < 2 \) are given in [11]. If \( t_j = 2 \) for \( j = 1, 2 \), then \( 2^*(t_j) = 2, j = 1, 2 \), and we have on the right hand side of equation (1.1) a sum of two Hardy potentials. In this situation we can look at (1.1) as an eigenvalue problem by replacing the right hand side of the equation by

\[
\lambda \left( \frac{P_1(x)}{|x|^2} + \frac{P_2(x)}{|x-\xi|^2} \right) u
\]

where \( \lambda \in \mathbb{R} \) is an eigenvalue parameter (see [12]). For elliptic problems involving the Hardy potential we also refer to papers [5], [13], [14], [19], [20], [21], [22], [23], [25], where further bibliographical references can be found.
The coefficients $P_j$, $j = 1, 2$, are assumed to be continuous on $\bar{\Omega}$. Further assumptions on $P_j$ will be formulated later. We look for solutions of problem (1.1) in a Sobolev space $H^1(\Omega)$ equipped with norm
\[ \|u\|^2 = \int_{\Omega} (|\nabla u|^2 + u^2) \, dx. \]

By $H^1_0(\Omega)$ we denote a Sobolev space obtained as the closure of the space $C^\infty_0(\Omega)$ with respect to the norm
\[ \|u\|_{H^1_0} = \int_{\Omega} |\nabla u|^2 \, dx. \]

Problems discussed in this paper are related to the optimal constant of the Hardy - Sobolev type. The best Hardy - Sobolev constant for the domain $\Omega \subset \mathbb{R}^N$ is defined by
\[ S_s^{H}(\Omega) = \inf_{f_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1, u \in H^1_0(\Omega)} \int_{\Omega} |\nabla u|^2 \, dx, \quad (1.2) \]
where $2^*(s) = \frac{2(N-s)}{N-2}$, $0 \leq s \leq 2$. If $\Omega = \mathbb{R}^N$, we write $S_s^{H}$ instead of $S_s^{H}(\Omega)$. If $s = 0$, then $S_0^{H}(\Omega) = S$ is the best Sobolev constant which is independent of $\Omega$. In the case $0 < s < 2$, $S_s^{H}(\Omega)$ depends on $\Omega$ (see [17], [18]). If $0 \leq s < 2$, then $S_s^{H}$ is attained by a family of functions
\[ W_s^\epsilon(x) = \frac{C_N \epsilon^{\frac{N-2}{2-s}}}{(\epsilon^2 + |x|^{2-s})^{\frac{N-2}{2-s}}}, \quad \epsilon > 0, \quad (1.3) \]
where $C_N$ is a normalizing positive constant depending on $N$ and $s$. Obviously, $W_s^\epsilon$ satisfies the equation
\[ -\Delta u = \frac{|u|^{2^*(s)-1}}{|x|^s} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}. \]

We also have
\[ \int_{\mathbb{R}^N} |\nabla W_s^\epsilon|^2 \, dx = \int_{\mathbb{R}^N} \frac{(W_s^\epsilon)^{2^*(s)}}{|x|^s} \, dx = (S_s^{H})^{\frac{2}{2-s}}. \]

From the definition of the Hardy - Sobolev constant $S_s^{H}(\Omega)$ it follows
\[ S_s^{H}(\Omega) \left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^*(s)}} \leq \int_{\Omega} |\nabla u|^2 \, dx \]
for every $u \in H^1_0(\Omega)$. We need an extension of this inequality to the space $H^1(\Omega)$ (see [10]).
Lemma 1.1. Let $0 \in \Omega$. Then there exists a constant $K > 0$ such that
\[
\left( \int_{\Omega} \frac{|u|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2^*(s)}{2^*(s)-s}} \leq K \int_{\Omega} (|\nabla u|^2 + u^2) \, dx \tag{1.4}
\]
for every $u \in H^1(\Omega)$.

A solution $u \in H^1(\Omega)$ of (1.1) is understood in a distributional (or weak) sense, that is,
\[
\int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)-2} uv \, dx + \int_{\Omega} \frac{P_2(x)}{|x-\xi|^{t_2}} |u|^{2^*(t_2)-2} uv \, dx
\]
for every $v \in H^1(\Omega)$. If $u \in H^1(\Omega)$ is a solution of (1.1) then
\[
0 = \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)-1} \, dx + \int_{\Omega} \frac{P_2(x)}{|x-\xi|^{t_2}} |u|^{2^*(t_2)-1} \, dx
\]
So if $P_1$ and $P_2$ are nonnegative and at least one of them not identically equal to 0, then problem (1.1) does not have a solution. Therefore, we assume
\[(P) \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx < \infty, P_1 \text{ changes sign and } P_2(x) > 0 \text{ on } \bar{\Omega}.
\]
We use the decomposition of the space $H^1(\Omega)$
\[H^1(\Omega) = V \oplus \mathbb{R}, \ V = \{v \in H^1(\Omega) \mid \int_{\Omega} v(x) \, dx = 0\}.
\]
This decomposition yields the following equivalent norm on $H^1(\Omega)$
\[
\|u\|_V^2 = \|\nabla u\|_2^2 + t^2, \ v \in V, \ t \in \mathbb{R}.
\]
We note that inequality (1.4) in the space $V$ takes the form: there exists a constant $K_1 > 0$ such that
\[
\left( \int_{\Omega} \frac{|v|^{2^*(s)}}{|x|^s} \, dx \right)^{\frac{2^*(s)}{2^*(s)-s}} \leq K_1 \int_{\Omega} |\nabla v|^2 \, dx
\]
for every $v \in V$.

We frequently use in this paper the following qualitative property:
\[(S) \text{ there exists a constant } \eta > 0 \text{ such that for every } t \in \mathbb{R} \text{ and } v \in V \text{ the inequality}
\[
\left( \int_{\Omega} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \leq \eta |t|
\]
yields
\[
\int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |v + t|^{2^*(t_1)} \, dx \leq \frac{|t|^{2^*(t_1)}}{2} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx.
\]
This follows from the continuity of the embedding of $H^1(\Omega)$ into the space $L^{2^* (t_1)}(\Omega, \frac{1}{|x|^{t_1}})$ (see also [3]). Solutions of problem (1.1) will be obtained as critical points of the variational functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^* (t_1)} \int_\Omega \frac{P_1(x)}{|x|^{t_1}} |u|^{2^* (t_1)} \, dx$$

$$- \frac{1}{2^* (t_2)} \int_\Omega \frac{P_2(x)}{|x|^{t_2}} |u|^{2^* (t_2)} \, dx.$$

To study problem (1.1) we distinguish three cases: (i) $2^* (t_1) < 2^* (t_2)$, (ii) $2^* (t_1) = 2^* (t_2)$ and (iii) $2^* (t_1) > 2^* (t_2)$. In the cases (i) and (ii) solutions are obtained via the mountain-pass principle. In the case (iii) we use a local minimization.

The paper is organized as follows. Sections 2 and 3 are devoted to the study of Palais-Smale sequences. In the final Section 4 we present the existence theorems for problem (1.1).

Throughout this paper, in a given Banach space we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, are denoted by $\| \cdot \|_p$.

2. The mountain-pass geometry and (PS) sequences of $J$

We recall that a $C^1$ functional $\phi : X \to \mathbb{R}$ on a Banach space $X$ satisfies the Palais-Smale condition at level $c$ ((PS)$_c$ condition for short), if each sequence $\{x_n\} \subset X$ such that (*) $\phi(x_n) \to c$ and (**) $\phi'(x_n) \to 0$ in $X^*$ is relatively compact in $X$. Finally, any sequence $\{x_n\}$ satisfying (*) and (**) is called a Palais-Smale sequence at level $c$ (a (PS)$_c$ sequence for short).

We distinguish three cases: (i) $2^* (t_1) < 2^* (t_2)$, (ii) $2^* (t_1) = 2^* (t_2)$ and (iii) $2^* (t_1) > 2^* (t_2)$.

We begin with the case $2^* (t_1) < 2^* (t_2)$.

**Proposition 2.1.** Suppose that (P) and $2^* (t_1) < 2^* (t_2)$ hold. Then every (PS)$_c$ sequence is bounded.

**Proof.** Let $\{u_n\} \subset H^1(\Omega)$ be a (PS)$_c$ sequence. We have

$$J(u_n) - \frac{1}{2^* (t_1)} \langle J(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{2^* (t_1)} \right) \int_\Omega |\nabla u_n|^2 \, dx$$

$$+ \left( \frac{1}{2^* (t_1)} - \frac{1}{2^* (t_2)} \right) \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} |u_n|^{2^* (t_2)} \, dx = c + o(1) + \epsilon_n \|u_n\|.$$

where $\epsilon_n \to 0$. From this we deduce that there exists a constant $C > 0$ such that
\[ \int_{\Omega} |\nabla u_n|^2 \, dx, \quad \int_{\Omega} \frac{P_2(x)}{|x - \xi|^2} |u_n|^{2^*(t_2)} \, dx \leq C(1 + \|u_n\|) \] (2.1)
for every $n$. Let $d = \text{diam } \Omega$ and $\tilde{m} = \min_{x \in \Omega} P_2(x)$. Then
\[ \frac{\tilde{m}}{d} \int_{\Omega} |u_n|^{2^*(t_2)} \, dx \leq \int_{\Omega} \frac{P_2(x)}{|x - \xi|^2} |u_n|^{2^*(t_2)} \, dx \leq C(1 + \|u_n\|). \]

By the Hölder inequality we deduce from this
\[ \int_{\Omega} u_n^2 \, dx \leq |\Omega|^{1 - \frac{2}{2^*(t_2)}} \left( \int_{\Omega} |u_n|^{2^*(t_2)} \, dx \right)^{\frac{2}{2^*(t_2)}} \leq \tilde{C}|\Omega|^{1 - \frac{2}{2^*(t_2)}} \left( 1 + \|u_n\|^{\frac{2}{2^*(t_2)}} \right), \] (2.2)
where $\tilde{C}$ is a constant independent of $n$. Inequalities (2.1) and (2.2) yield the boundedness of $\{u_n\}$ in $H^1(\Omega)$.

**Proposition 2.2.** Suppose that $\text{(P)}$ and $2^*(t_1) < 2^*(t_2)$ hold. Then there exist constants $\kappa > 0$ and $\rho > 0$ such that
\[ J(u) \geq \kappa \quad \text{for } \|u\| = \rho. \]

**Proof.** We use property (S). We distinguish two cases (i) $\|\nabla v\|_2 \leq \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$, where $\eta > 0$ is a constant from property (S) and $u = v + t$, $v \in V$, $t \in \mathbb{R}$. If (i) holds and $\|\nabla v\|_2^2 + t^2 = \rho^2$, then $t^2 \geq \frac{\rho^2}{1 + \eta^2}$. By (S) we get
\[ \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx \leq \frac{|t|^{2^*(t_1)}}{2} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx = -|t|^{2^*(t_1)} \alpha, \]
where $\alpha = -\frac{1}{2} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx > 0$. From this we derive the estimate of $J$ from below
\[ J(u) \geq \frac{\alpha \rho^{2^*(t_1)}}{2^*(t_1)(1 + \eta^2)} - \frac{1}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^{t_2}} |u|^{2^*(t_2)} \, dx. \] (2.3)

In the case (ii) we have
\[ \|u\|_V \leq \|\nabla v\|_2 \left( 1 + \frac{1}{\eta^2} \right)^{\frac{1}{2}}. \] (2.4)

It follows from Lemma 1.1 that
\[ \left| \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx \right| \leq C_1 \|u\|_V^{2^*(t_1)} \leq C_1 \|\nabla v\|_2^{2^*(t_1)} \left( 1 + \frac{1}{\eta^2} \right)^{\frac{2^*(t_1)}{2}} \]
for some constant $C_1 > 0$. Thus
\[ J(u) \geq \frac{1}{2} \| \nabla v \|_2^2 - C_1 \| \nabla v \|_2^{2^*(t_1)} (1 + \frac{1}{\eta^2})^{\frac{2^*(t_1)}{2}} - \frac{1}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^t_2} |u|^{2^*(t_2)} \, dx. \]
Taking $\| \nabla v \|_2^2 \leq \rho^2$ small enough we derive from the above inequality that
\[ J(u) \geq \frac{1}{4} \| \nabla v \|_2^2 - \frac{1}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^t_2} |u|^{2^*(t_2)} \, dx. \]
If $\|u\|_V = \rho$, then combining (2.4) with the last inequality we get
\[ J(u) \geq \frac{\rho^2 \eta^2}{4(1 + \eta^2)} - \frac{1}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^t_2} |u|^{2^*(t_2)} \, dx. \]
Estimates (2.3) and (2.5) yield
\[ J(u) \geq \min \left( \frac{\rho^2 \eta^2}{4(1 + \eta^2)}, \frac{\alpha \rho^{2^*(t_1)}}{2^*(t_1)(1 + \eta^2)^{\frac{2^*(t_1)}{2}}} \right) - \frac{1}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^t_2} |u|^{2^*(t_2)} \, dx. \]
Applying Lemma 1.1 to the integral on the right hand side gives
\[ J(u) \geq \min \left( \frac{\rho^2 \eta^2}{4(1 + \eta^2)}, \frac{\alpha \rho^{2^*(t_1)}}{2^*(t_1)(1 + \eta^2)^{\frac{2^*(t_1)}{2}}} \right) - C_2 \rho^{2^*(t_2)} \]
for some constant $C_2 > 0$. Since $2 < 2^*(t_1) < 2^*(t_2)$, taking $\rho > 0$ sufficiently small we can find a constant $\kappa > 0$ such that
\[ J(u) \geq \kappa \text{ for } \|u\|_V = \rho \]
which completes the proof.

We now turn our attention to the case $2^*(t_1) = 2^*(t_2)$.

**Proposition 2.3.** Suppose that $(P)$ and $2^*(t_1) = 2^*(t_2)$ hold. Moreover, assume that
\[ \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx + \int_{\Omega} \frac{P_2(x)}{|x - \xi|^t_2} \, dx \neq 0. \]
Then $(PS)_c$ sequences of $J$ are bounded in $H^1(\Omega)$.

**Proof.** Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence. We use the decomposition $u_n = v_n + t_n$, $v_n \in V$ and $t_n \in \mathbb{R}$. First we show that $\{t_n\}$ is bounded. Arguing by contradiction, assume $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated in a similar way). We have
\[ c + o(1) + \varepsilon_n \|u_n\| = J(u_n) - \frac{1}{2^*(t_1)} \langle J'(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \int_{\Omega} |\nabla v_n|^2 \, dx, \]
with $\epsilon_n \to 0$. This shows that
\[
\|\nabla v_n\|_2^2 \leq C \left( 1 + \|u_n\|_V \right)
\]
for some constant $C > 0$ independent of $n$. Inequality (2.6) can be rewritten in the following form
\[
\|\nabla (\frac{v_n}{t_n})\|_2^2 \leq C \left( \frac{1}{t_n} + \left[ \int_{\Omega} |\nabla (\frac{v_n}{t_n})|^2 \, dx + 1 \right] \right)^{1/2}.
\]
Hence $\|\nabla (\frac{v_n}{t_n})\|_2^2 \to 0$ and consequently $\frac{v_n}{t_n} \to 0$ in $L^{2^*(t_1)}(\Omega, \frac{1}{|x|^{t_1}})$ and $L^{2^*(t_1)}(\Omega, \frac{1}{|x-\xi|^{t_1}})$. On the other hand we have
\[
c + o(1) + \epsilon_n \|u_n\|_V = J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle = \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \left( \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u_n|^{2^*(t_1)} \, dx \right)
\]
\[
+ \frac{\int_{\Omega} P_2(x)}{|x-\xi|^{t_1}} |u_n|^{2^*(t_1)} \, dx \right).
\]
Dividing this equality by $t_n^{2^*(t_1)}$ we get
\[
\frac{1}{t_n^{2^*(t_1)}} \left( c + o(1) + \epsilon_n \|u_n\|_V \right)
\]
\[
= \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \left( \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \frac{v_n}{t_n} + 1 \right)^{2^*(t_1)} \, dx
\]
\[
+ \frac{\int_{\Omega} P_2(x)}{|x-\xi|^{t_1}} \frac{v_n}{t_n} + 1 \right)^{2^*(t_1)} \, dx \right).
\]
Letting $n \to \infty$ we obtain
\[
\int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx + \int_{\Omega} \frac{P_2(x)}{|x-\xi|^{t_2}} \, dx = 0
\]
and we have arrived at a contradiction. Since $\{t_n\}$ is bounded, it follows from (2.6) that $\|\nabla v_n\|_2$ is also bounded and the result follows. \(\square\)

In the case $2^*(t_1) = 2^*(t_2)$ we can obtain the mountain-pass geometry for a modified variational functional
\[
J_\mu(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{2^*(t_1)} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx
\]
\[
- \frac{\mu}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x|^{t_2}} |u|^{2^*(t_2)} \, dx,
\]
where $0 < \mu < \mu_0$ is a parameter and $\mu_0 > 0$ is sufficiently small. The variational functional $J_\mu$ corresponds to the following Neumann problem

$$
\begin{aligned}
-\Delta u &= P_1(x)\frac{|u|^{2^*(t_1)-2}u + \mu P_2(x)\frac{|u|^{2^*(t_1)-2}u}{|x-\xi|^{t_1}}}{|x|^{t_1}} \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \quad u > 0 \quad \text{on } \Omega.
\end{aligned}
$$

(2.7)

**Proposition 2.4.** Suppose that (P) and $2^*(t_1) = 2^*(t_2)$ hold. Then there exist constants $\mu_0 > 0$, $\kappa > 0$ and $\rho > 0$ such that

$$J_\mu(u) \geq \kappa \quad \text{for } \|u\| = \rho$$

and $0 < \mu < \mu_0$.

**Proof.** As in the proof of Proposition 2.2 we get

$$J_\mu(u) \geq \min\left(\frac{\rho^2\eta^2}{4(1+\eta^2)}, \frac{\alpha \rho^{2^*(t_1)}}{2^*(t_1)(1+\eta^2)^{2^*(t_1)/2}}\right) - \frac{\mu}{2^*(t_1)} \int_\Omega \frac{P_2(x)|u|^{2^*(t_1)}}{|x-\xi|^{t_1}} \ dx.$$ 

It then follows from Lemma 1.1 that

$$J_\mu(u) \geq \min\left(\frac{\rho^2\eta^2}{4(1+\eta^2)}, \frac{\alpha \rho^{2^*(t_1)}}{2^*(t_1)(1+\eta^2)^{2^*(t_1)/2}}\right) - \mu C_2 \rho^{2^*(t_1)},$$

for some positive constant $C_2 > 0$. The result follows by taking $\mu_0$ sufficiently small.

Problem (2.7) does not have a solution for $\mu$ large.

**Proposition 2.5.** Suppose that assumptions of Proposition 2.4 hold. Then problem (2.7) does not admit a solution for

$$\mu > -\frac{\int_\Omega \frac{P_1(x)}{|x|^{t_1}} \ dx}{\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}} \ dx}.$$ 

(2.8)

**Proof.** Suppose that $u$ is a solution of problem (2.7). Let $\epsilon > 0$. Testing (2.7) with $\phi(x) = (u^2 + \epsilon^2)^{-\frac{2^*(t_1)-1}{2}}$ we get

$$
0 > -\left(2^*(t_1) - 1\right) \int_\Omega |\nabla u|^2 u (u^2 + \epsilon^2)^{-\frac{2^*(t_1)+1}{2}} \ dx
= \int_\Omega \frac{P_1(x)}{|x|^{t_1}} \frac{|u|^{2^*(t_1)-1}}{(u^2 + \epsilon^2)^{\frac{2^*(t_1)+1}{2}}} \ dx
+ \mu \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}} \frac{|u|^{2^*(t_1)-1}}{(u^2 + \epsilon^2)^{\frac{2^*(t_1)-1}{2}}} \ dx.
$$
Hence
\[
\mu \int_{\Omega} P_2(x) \frac{u^{2^*(t_1)-1}}{(u^2 + \epsilon^2)^{2^*(t_1)/2-1}} dx < - \int_{\Omega} P_1(x) \frac{u^{2^*(t_1)-1}}{(u^2 + \epsilon^2)^{2^*(t_1)/2-1}} dx.
\]
Letting \( \epsilon \to 0 \) we obtain
\[
\mu \leq - \frac{\int_{\Omega} P_1(x)}{\int_{\Omega} P_2(x)} dx
\]
and the result follows. \( \square \)

**Remark 2.1.** It is clear that problem (2.7) has no solution if
\[
\frac{P_1(x)}{|x|^{t_1}} + \mu \frac{P_2(x)}{|x - \xi|^{t_1}} > 0 \quad \text{on} \quad \Omega. \tag{2.9}
\]
Obviously inequality (2.9) yields (2.8).

Finally, we consider the case \( 2^*(t_1) > 2^*(t_2) \). As in the case \( 2^*(t_1) = 2^*(t_2) \) we consider the nonlinear Neumann problem involving a parameter \( \mu > 0 \)
\[
\begin{cases}
-\Delta u = \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)-2} u + \mu \frac{P_2(x)}{|x - \xi|^{t_1}} |u|^{2^*(t_2)-2} u & \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial \Omega, \\
u > 0 & \text{on} \quad \Omega, 
\end{cases} \tag{2.10}
\]
where \( 0 < \mu < \mu_* \) with \( \mu_* > 0 \) small. Let
\[
I_\mu(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2^*(t_1)} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} dx
- \frac{\mu}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^{t_2}} |u|^{2^*(t_2)} dx.
\]

**Proposition 2.6.** Suppose \((P)\) and \( 2^*(t_1) > 2^*(t_2) \) hold. Then there exist constants \( \mu_* > 0, \kappa > 0 \) and \( \rho > 0 \) such that
\[
I_\mu(u) \geq \kappa \quad \text{for} \quad \|u\| = \rho \tag{2.11}
\]
and \( 0 < \mu < \mu_* \). Moreover,
\[
\inf_{\|u\| \leq \rho} I_\mu(u) < 0 \quad \text{for} \quad 0 < \mu < \mu_*.
\]

**Proof.** The proof of the first part is similar to that of Proposition 2.2. To show the second part observe that for a constant \( t > 0 \) we have
\[
I_\mu(t) = - \frac{t^{2^*(t_1)}}{2^*(t_1)} \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} dx - \frac{\mu t^{2^*(t_2)}}{2^*(t_2)} \int_{\Omega} \frac{P_2(x)}{|x - \xi|^{t_2}} dx.
\]
Since \( 2^*(t_1) > 2^*(t_2) \), \( I_\mu(t) < 0 \) for \( t > 0 \) sufficiently small. \( \square \)
3. Palais - Smale condition

We commence with the case $2^*(t_1) < 2^*(t_2)$.

**Proposition 3.1.** Let $0, \xi \in \partial \Omega$. Suppose that $(P)$ and $2^*(t_1) < 2^*(t_2)$ hold. Moreover assume that $P_1(0) > 0$. Then $(PS)_c$ condition is satisfied for

$$c < c^* := \min \left( \frac{(2-t_1) \left( S_H^{t_1} \right)^{\frac{N-t_1}{2-t_1}}}{4(N-t_1) P_1(0)^{\frac{N-t_1}{2-t_1}}}, \frac{(2-t_2) \left( S_H^{t_2} \right)^{\frac{N-t_2}{2-t_2}}}{4(N-t_2) P_1(\xi)^{\frac{N-t_2}{2-t_2}}} \right) \quad (3.1)$$

**Proof.** Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence with $c$ satisfying (3.1). By Proposition 2.1 $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega), L^{2^*(t_1)}(\Omega, \frac{1}{|x|^t_1})$ and $L^{2^*(t_2)}(\Omega, \frac{1}{|x-\xi|^t_2})$. It follows from P.L. Lions’ concentration - compactness principle (see [24]) that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + b_0 \delta_0 + b_\xi \delta_\xi,$$

and

$$\frac{|u_n|^{2^*(t_1)}}{|x|^{t_1}} \rightharpoonup \frac{|u|^{2^*(t_1)}}{|x|^{t_1}} + a_0 \delta_0,$$

in the sense of measure, where $b_0, b_\xi, a_0, a_\xi$ are nonnegative constants and $\delta_0$ and $\delta_\xi$ denote the Dirac measures assigned to 0 and $\xi$, respectively. The constants $b_0, b_\xi, a_0, a_\xi$ satisfy inequalities

$$\frac{a_0^{2^*(t_1)} S_H^{t_1}}{2^{\frac{N-t_1}{2-t_1}}} \leq b_0 \quad \text{and} \quad \frac{a_\xi^{2^*(t_2)} S_H^{t_2}}{2^{\frac{N-t_2}{2-t_2}}} \leq b_\xi. \quad (3.2)$$

We have

$$c = \lim_{n \to \infty} \left( J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \right) \quad (3.3)$$

$$= \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \int_\Omega \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx$$

$$+ \left( \frac{1}{2} - \frac{1}{2^*(t_2)} \right) \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} |u|^{2^*(t_2)} \, dx$$

$$+ \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) a_0 P_1(0) + \left( \frac{1}{2} - \frac{1}{2^*(t_2)} \right) a_\xi P_2(\xi).$$

We now observe that

$$0 \leq \int_\Omega |\nabla u|^2 \, dx = \int_\Omega \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} |u|^{2^*(t_2)} \, dx.$$
Hence we derive from (3.3) that
\[
    c \geq \left( \frac{1}{2} - \frac{1}{2^s(t_1)} \right) \left( \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^s(t_1)} \, dx \right) + \int_{\Omega} \frac{P_2(x)}{|x-\xi|^{t_2}} |u|^{2^s(t_2)} \, dx \\
    + \left( \frac{1}{2} - \frac{1}{2^s(t_1)} \right) a_o P_1(0) + \left( \frac{1}{2} - \frac{1}{2^s(t_2)} \right) a_{\xi} P_2(\xi) \\
    \geq \left( \frac{1}{2} - \frac{1}{2^s(t_1)} \right) a_o P_1(0) + \left( \frac{1}{2} - \frac{1}{2^s(t_2)} \right) a_{\xi} P_2(\xi).
\]

Let \( \varphi_\delta, \delta > 0 \), be a family of \( C^1 \)-functions concentrating at 0 as \( \delta \to 0 \). We derive from \( \langle J'(u_n), u_n \varphi_\delta^2 \rangle \to 0 \) that
\[
    b_o \leq P_1(0) a_o \quad \text{and} \quad b_{\xi} \leq P_2(\xi) a_{\xi}.
\]

To complete the proof it is sufficient to show that \( a_o = a_{\xi} = 0 \). Assume that \( a_o > 0 \), then (3.2) and (3.5) imply that
\[
    a_o \geq \frac{(S_H^{t_1})^{\frac{N-t_1}{2-t_1}}}{2P_1(0)^{\frac{N-2}{2-t_1}}}.
\]

It then follows from (3.4) that
\[
    c \geq \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2^s(t_1)} \right) \left( S_H^{t_1} \right)^{\frac{N-t_1}{2-t_1}} \frac{N-2}{2-t_1} P_1(0)^{\frac{N-2}{2-t_1}},
\]
which is impossible. So \( a_o = 0 \). In a similar manner we show that one has \( a_{\xi} = 0 \).

\[ \square \]

**Remark 3.1.** Inspection of the proof of Proposition 3.1 shows that if \( P(0) \leq 0 \), the \((PS)_c\) sequence cannot concentrate at 0. In this case (3.1) takes the form
\[
    c < c^* = \frac{(2-t_2)}{4(N-t_2)} \left( S_H^{t_2} \right)^{\frac{N-t_2}{2-t_2}} P_1(\xi)^{\frac{N-2}{2-t_2}}.
\]

We now consider the case \( 2^s(t_1) = 2^s(t_2) \).

**Proposition 3.2.** Let \( 0, \xi \in \partial \Omega \). Let \((P)\) and \( 2^s(t_1) = 2^s(t_2) \) hold. Suppose that \( P_1(0) > 0 \), \( 0 < \mu \) and
\[
    \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} \, dx + \mu \int_{\Omega} \frac{P_2(x)}{|x-\xi|^{t_1}} \, dx \neq 0.
\]
Then \( J_\mu \) satisfies the \((PS)_c\) condition for
\[
c < \tilde{c} := \min\left( \frac{(2 - t_1)}{4(N - t_1)} \left( \frac{S_{H}^{t_1}}{P_1(0)^{N - t_1}} \right), \frac{(2 - t_2)}{4(N - t_2)} \left( \frac{S_{H}^{t_2}}{(\mu P_2(\xi))^{N - t_2}} \right) \right)
\]

**Proof.** We argue as in the proof of Proposition 3.1. Let \( \{u_n\} \subset H^1(\Omega) \) be a \((PS)_c\) sequence for \( J \). By Proposition 2.3 \( \{u_n\} \) is bounded in \( H^1(\Omega) \). So we may assume that \( u_n \rightharpoonup u \) in \( H^1(\Omega) \) and \( L^{2^*(t_1)}(\Omega, \frac{1}{|x|^{t_1}}) \). We have

\[
c = \lim_{n \to \infty} \left( J(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \right) = \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \left( \int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx \right)
\]  
+ \mu \int_{\Omega} \frac{P_2(x)}{|x - \xi|^{t_1}} |u|^{2^*(t_1)} \, dx) 
\]  
+ \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) (a_\xi P_1(0) + \mu a_\xi P_2(\xi)),
\]

where \( b_\xi, a_\xi \) and \( a_\xi \) satisfy
\[
b_\xi \leq P_1(0)a_\xi \quad \text{and} \quad b_\xi \leq \mu P_2(\xi)a_\xi.
\]

We now observe that
\[
\int_{\Omega} \frac{P_1(x)}{|x|^{t_1}} |u|^{2^*(t_1)} \, dx + \mu \int_{\Omega} \frac{P_2(x)}{|x - \xi|^{t_1}} |u|^{2^*(t_1)} \, dx \geq 0.
\]

If \( a_\xi > 0 \), then
\[
c > \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \left( \frac{S_{H}^{t_1}}{P_1(0)^{N - t_1}} \right),
\]
which is impossible. Similarly, if \( a_\xi > 0 \), then
\[
c > \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2^*(t_1)} \right) \left( \frac{S_{H}^{t_2}}{(\mu P_2(\xi))^{N - t_2}} \right),
\]
which again gives a contradiction and the result follows. \( \square \)

**Proposition 3.3.** Let \( 0, \xi \in \partial \Omega \). Suppose that \((P)\) and \( 2^*(t_1) > 2^*(t_2) \) hold. Moreover, assume that \( P_1(0) > 0 \) and \( 0 < \mu < \mu_\ast \). If \( \{u_n\} \) is a bounded in \( H^1(\Omega) \) a \((PS)_c\) sequence for the functional \( I_\mu \) with
\[
c < \min\left( \frac{(2 - t_1)}{4(N - t_1)} \left( \frac{S_{H}^{t_1}}{P_1(0)^{N - t_1}} \right), \frac{(2 - t_2)}{4(N - t_2)} \left( \frac{S_{H}^{t_2}}{(\mu P_2(\xi))^{N - t_2}} \right) \right),
\]
then \( \{u_n\} \) contains a subsequence converging weakly to nonzero solution of (2.10).
Denoting by $H$ asymptotic estimate for $I$ where $r$.

We choose $r = 0$ so that $P_1(x) > 0$ on $B(0, 2r_0) \subset \Omega$. Let $\phi$ be a $C^1$-function such that $\phi(x) = 1$ on $B(0, r_0)$, $\phi(x) = 0$ on $\mathbb{R}^N - B(0, 2r_0)$ and $0 \leq \phi(x) \leq 1$ on $\mathbb{R}^N$. To estimate the mountain-pass level of the functional $J$ we use the function given by (1.3) with $s = t_1$. Let $w_{\epsilon,t_1}(x) = \phi(x)W_{\epsilon,t_1}^1(x)$ and define a function $I$ on $H^1(\Omega)$ by

$$I(u) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\left(\int_\Omega \left|\frac{u^{2^*(t_1)}}{|x|^{t_1}}\right|^\frac{N-2}{N-t_1} \, dx\right)^\frac{N-2}{N-t_1}}.$$  

Denoting by $H(0)$ a mean curvature of $\partial \Omega$ at 0, we have the following asymptotic estimate for $I(w_{\epsilon,t_1})$ (see [11], [17]):

$$I(w_{\epsilon,t_1}) = \begin{cases} 
\frac{s_{t_1}^N}{2^{\frac{N-2}{N-t_1}}} - H(0)a_N \epsilon^{2^{\frac{2}{N-t_1}}} + o( \epsilon^{2^{\frac{2}{N-t_1}}}) & , \ N \geq 4 \\
\frac{s_{t_1}^N}{2^{\frac{N-2}{N-t_1}}} - H(0)b_N \epsilon^{2^{\frac{2}{N-t_1}}} |\ln \epsilon| + o( \epsilon^{2^{\frac{2}{N-t_1}}}) & , \ N = 3,
\end{cases}$$

where $a_N$ and $b_N$ are positive constants.

**Theorem 4.1.** Let $0, \xi \in \partial \Omega$ and $H(0) > 0$. Suppose (P), $2^*(t_1) < 2^*(t_2)$ and (4.1) hold. If

$$|P_1(x) - P_1(0)| = o( |x|^{2^{\frac{2}{N-t_1}}})$$

for $x$ close to 0, then problem (1.1) admits a solution.

**Proof.** By Proposition 2.2, the functional $J$ has a mountain-pass structure. Since $2^*(t_1) < 2^*(t_2)$ there exists a function $v \in H^1(\Omega)$ such that $\|v\| > \rho$ and $J(v) < 0$. Let $c$ be a mountain-pass level for $J$, that is,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$
where
\[ \Gamma = \{ \gamma \in C([0, 1], H^1(\Omega)), \gamma(0) = 0, \gamma(1) = v \}, \]
where \( v = tw_{t,t_1} \) with \( t > 0 \) sufficiently large. It is clear that
\[ c \leq \max_{t \geq 0} J(tw_{t,t_1}) \leq \max_{t \geq 0} \left( \frac{t^2}{2} \int_\Omega |\nabla w_{t,t_1}|^2 \, dx \right) \]
\[ - \frac{t^{2^*(t_1)}}{2^*(t_1)} \int_\Omega \frac{P_1(x)}{|x|^{t_1}} |w_{t,t_1}|^{2^*(t_1)} \, dx \]
\[ = \frac{(2 - t_1)}{2(N - t_1)} \left( \int_\Omega |\nabla w_{t,t_1}|^2 \, dx \right)^{\frac{N - t_1}{2}}. \]

Obviously, the curve \( \gamma(s) = stw_{t,t_1}, 0 \leq s \leq 1, \) with \( t \) sufficiently large, belongs to \( \Gamma \). We now observe that
\[ \int_\Omega \frac{P_1(x)}{|x|^{t_1}} |w_{t,t_1}|^{2^*(t_1)} \, dx \]
\[ = P_1(0) \int_\Omega \frac{|w_{t,t_1}|^{2^*(t_1)}}{|x|^{t_1}} \, dx + o\left( \epsilon^{\frac{2}{2 - t_1}} \right). \]

Combining (4.2), (4.3) and (4.4) we derive \( c < c^* \). Thus by Proposition 3.1 the functional \( J \) satisfies the (PS) condition at the level \( c \). The existence of a solution \( u \neq 0 \) of (1.1) follows from the mountain-pass principle. By Theorem 10 in [3] we may assume that \( u \geq 0 \) on \( \Omega \). The fact that \( u > 0 \) on \( \Omega \) follows from Harnack inequality (see [16]).

Similarly, we have

**Theorem 4.2.** Let \( 0, \xi \in \partial \Omega, H(\xi) > 0 \). Suppose that (P), \( 2^*(t_1) < 2^*(t_2) \) and
\[ c^* = \frac{(2 - t_2)}{4(N - t_2)} \left( \frac{S_{t_2}^H}{P_2(\xi)} \right)^{\frac{N - t_2}{2 - t_2}} \quad \text{and} \quad P_1(0) > 0. \]

If
\[ |P_2(\xi) - P_2(x)| = o\left( |x|^{\frac{2}{2 - t_2}} \right) \]
for \( x \) close to \( \xi \), then problem (1.1) admits a solution.

We now consider the case \( 2^*(t_1) = 2^*(t_2) \). We can always assume that \( 0 < \mu < \mu_0 < \frac{P_1(0)}{P_2(\xi)} \) by taking \( \mu_0 \) smaller if necessary. Then
\[ c^* = \frac{(2 - t_1)}{4(N - t_1)} \left( \frac{S_{t_1}^H}{P_1(0)} \right)^{\frac{N - t_1}{2 - t_1}}. \]

Propositions 2.3, 2.4, 3.2 and Remark 2.1 lead to the following existence theorem in the case \( 2^*(t_1) = 2^*(t_2) \).
Theorem 4.3. Let $0, \xi \in \partial \Omega$. Let $P_1(0) > 0$, $2^*(t_1) = 2^*(t_2)$, $H(0) > 0$, $0 < \mu < \mu_*$ and
\[
\int_\Omega \frac{P_1(x)}{|x|^{t_1}} \, dx + \mu \int_\Omega \frac{P_2(x)}{|x - \xi|^{t_1}} \, dx < 0.
\]
Moreover assume that (P) holds and that
\[
|P_1(x) - P_1(0)| = o\left(\frac{|x|^{2}}{|x|^{2-t_1}}\right)
\]
for $x$ close to $0$, then problem (2.7) admits a solution.

Finally, in the case $2^*(t_1) > 2^*(t_2)$, by Proposition 2.6, the functional $I_\mu$ satisfies (2.11) and $\inf_{\|u\|_p} I_\mu(u) < 0$. Therefore we can apply the Ekeland variational principle and obtain the $(PS)_c$ sequence with $c = \inf_{\|u\|_p} I_\mu(u) < 0$ for $0 < \mu < \mu^*$. This sequence, according to Proposition 3.3, contains a subsequence weakly converging to nonzero solution of (2.10). This allows us to formulate the following existence result for problem (2.10):

Theorem 4.4. Let $0, \xi \in \partial \Omega$. Suppose (P), $2^*(t_1) > 2^*(t_2)$ and $P_1(0) > 0$ hold. Then problem (2.10) admits a solution.

Theorems 4.3 and 4.4 continue to hold for $\mu = 0$, that is, for the following problem
\[
\begin{cases}
-\Delta u = \frac{P(x)}{|x|^s}|u|^{2^*(s)-2} u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \ u > 0 \text{ on } \Omega,
\end{cases}
\]
where $0 < s < 2$ and $P(x)$ is a continuous function on $\bar{\Omega}$. Moreover, we assume that

(R) The function $P(x)$ changes sign and $\int_\Omega \frac{P(x)}{|x|^s} \, dx < 0$.

The corresponding variational functional is given by
\[
I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*(s)} \int_\Omega \frac{P(x)}{|x|^2} |u|^{2^*(s)} \, dx.
\]
Repeating the arguments from Sections 2 and 3 we can show that $I$ has a mountain - pass geometry. If $P(0) > 0$, then the $(PS)_c$ condition holds for
\[
c < \frac{(2 - s)}{4(N - s)} \left( \frac{S_H^{*}}{2-s} \right) \left( \frac{N-s}{P(0)} \right).
\]
If $P(0) \leq 0$, the $(PS)_c$ condition holds for every $c \in \mathbb{R}$. We can now state the following existence result for problem (4.5)
Theorem 4.5. Let $0 \in \partial \Omega$, $0 < s < 2$, $P(0) > 0$ and $H(0) > 0$. Moreover, assume that (R) holds and

$$|P(x) - P(0)| = o(|x|^2)$$

for $x$ close to 0. Then problem (4.5) admits a solution.

The proof is similar to that of Theorem 4.1 and is omitted.

Remark 4.1. In the case $2^*(t_1) < 2^*(t_2)$, that is $t_1 > t_2$, a solution $u$ of problem (1.1) satisfies the following estimate

$$\frac{\tilde{m}}{d^{t_1}} \int_\Omega u^{2(t_1-t_2) \frac{N-2}{N-2}} dx \leq \tilde{m} \int_\Omega \frac{u^{2(t_1-t_2) \frac{N-2}{N-2}}}{|x-\xi|^2} dx \leq \int_\Omega \frac{P_2(x)}{|x-\xi|^2} u^{2(t_1-t_2) \frac{N-2}{N-2}} dx \leq \int_\Omega \frac{P_1(x)}{|x|^2} dx,$$

where $\tilde{m} = \min_{x \in \Omega} P_2(x)$. Indeed, taking as a test function $\phi(x) = (u^2 + \epsilon^2)^{\frac{2^*(t_1)-1}{2}}$ (see the proof of Proposition 2.5) we get

$$0 \geq -(2^*(t_1) - 1) \int_\Omega |\nabla u|^2 u(u^2 + \epsilon^2)^{\frac{2^*(t_1)+1}{2}} dx = \int_\Omega \frac{P_1(x)}{|x|^2} \frac{u^{2^*(t_1)-1}}{(u^2 + \epsilon^2)^{\frac{2^*(t_1)-1}{2}}} dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^2} \frac{u^{2^*(t_2)-1}}{(u^2 + \epsilon^2)^{\frac{2^*(t_2)-1}{2}}} dx.$$

Letting $\epsilon \to 0$ the estimate follows. In a similar, way one can show that a solution $u$ of problem (2.10) (with $2^*(t_1) > 2^*(t_2)$) satisfies the following estimate

$$\frac{\tilde{m}}{d^{t_1}} \int_\Omega u^{2(t_1-t_2) \frac{N-2}{N-2}} dx \leq \tilde{m} \mu \int_\Omega \frac{u^{2(t_1-t_2) \frac{N-2}{N-2}}}{|x-\xi|^2} dx \leq \int_\Omega \frac{P_2(x)}{|x-\xi|^2} u^{2(t_1-t_2) \frac{N-2}{N-2}} dx \leq \int_\Omega \frac{P_1(x)}{|x|^2} dx.$$

References


On the Neumann problem involving the Hardy - Sobolev potentials


*Jan Chabrowski*

Department of Mathematics, The University of Queensland

St. Lucia 4072, Qld, Australia

E-mail: jhc@maths.uq.edu.au