Lattice preradicals versus module preradicals

TOMA ALBU AND MIHAI IOSIF

Communicated by Constantin Năstăescu
Dedicated to Professors Nicolae Dinculeanu and Solomon Marcus
in honour of their 90th birthdays

Abstract - This paper investigates the connections between lattice preradicals and module preradicals. We show that to any lattice preradical one associates in a canonical way a module preradical, but not conversely. However, to any module preradical, or more generally, to any preradical on a locally small Abelian category, we may associate a weaker form of a lattice preradical by introducing and investigating a class of subcategories, not necessarily full, of the category $\mathcal{LM}$ of all linear modular lattices, we call linearly closed.

Key words and phrases : Modular lattice, linear modular lattice, lattice preradical, weakly lattice preradical, module preradical, linearly closed subcategory, Abelian category, hereditary torsion theory.

Mathematics Subject Classification (2010) : 06C05, 06C99, 16S90, 18E10.

Introduction

The aim of this paper is to investigate the connections between the lattice preradicals introduced in [4] and the usual module preradicals.

Section 0 collects together some general notation and terminology on lattices, modules, and hereditary torsion theories needed in the sequel.

Section 1 presents some basic definitions and results of [3], [4], and [5] on linear morphisms of lattices and lattice preradicals.

In Section 2 we give the main results of the paper. Firstly, we show that any lattice preradical naturally induces a module preradical, or more generally, a preradical on any locally small Abelian category, but not conversely. Then, we introduce and investigate the concept of a linearly closed subcategory of the category $\mathcal{LM}$ of all linear modular lattices; these are subcategories of $\mathcal{LM}$ that are not necessarily full but enjoy some natural conditions that are in particular satisfied when considering subcategories of locally small Abelian categories or subcategories associated with $\tau$-saturated submodules with respect to a hereditary torsion theory $\tau$ on the category $\text{Mod-}R$ of all right $R$-modules over a unital ring $R$. 
Section 3 presents the more general concept of a preradical on a linearly closed subcategory of $\mathcal{L}\mathcal{M}$. Then, we show that we can naturally associate to preradicals on locally small Abelian categories and module categories equipped with hereditary torsion theories lattice preradicals on the linearly closed subcategories $\mathcal{S}\mathcal{C}_X$ and $\mathcal{S}\mathcal{C}_H$ discussed in Examples 2.7 and 2.9, respectively. In the final part of this section we show how the main results of [5] about lattice preradicals on $C_{11}$ lattices also hold for preradicals on linearly closed subcategories that are weakly hereditary.

0. Preliminaries

All lattices considered in this paper are assumed to be bounded, i.e., they have a least element denoted by $0$ and a greatest element denoted by $1$. Throughout this paper, $L$ will always denote such a lattice. We shall denote by $\mathcal{L}$ the class of all (bounded) lattices and by $\mathcal{M}$ the class of all (bounded) modular lattices.

For a lattice $L$ and elements $a \leq b$ in $L$ we write

$$b/a := [a, b] = \{ x \in L \mid a \leq x \leq b \}.$$ 

An initial interval of $b/a$ is any interval $c/a$ for some $c \in b/a$.

For all other undefined notation and terminology on lattices, the reader is referred to [1], [2], [7], and/or [8].

Throughout this paper $R$ will denote an associative ring with non-zero identity element, and Mod-$R$ (respectively, $R$-Mod) the category of all unital right (respectively, left) $R$-modules. The notation $M_R$ will be used to designate a unital right $R$-module $M$, and $N \leq M$ will mean that $N$ is a submodule of $M$. The lattice of all submodules of a module $M$ will be denoted by $\mathcal{L}(M)$.

A preradical on Mod-$R$ is a subfunctor $q$ of the identity functor $1_{\text{Mod-}R}$ of Mod-$R$. This means that $q$ assigns to each right $R$-module $M$ a submodule $q(M)$ of $M$ such that each morphism $f : M \to N$ in Mod-$R$ induces by restriction a morphism $q(f) : q(M) \to q(N)$, i.e., $f(q(M)) \leq q(N)$.

In this paper $\tau = (\mathcal{T}, \mathcal{F})$ will denote a fixed hereditary torsion theory on Mod-$R$ and $t_\tau(M)$ the $\tau$-torsion submodule of a right $R$-module $M$. It is well-known that the assignment $M \mapsto t_\tau(M)$, $M \in \text{Mod-}R$, defines a left exact (pre)radical on Mod-$R$. For any $M_R$ we shall denote

$$\text{Sat}_\tau(M) := \{ N \mid N \leq M \text{ and } M/N \in \mathcal{F} \},$$

and for any $N \leq M$ we shall denote by $\overline{N}$ the $\tau$-saturation of $N$ (in $M$) defined by $\overline{N}/N = t_\tau(M/N)$. The submodule $N$ is called $\tau$-saturated if $N = \overline{N}$. Note that

$$\text{Sat}_\tau(M) = \{ N \mid N \leq M, N = \overline{N} \},$$
so $\text{Sat}_\tau(M)$ is the set of all $\tau$-saturated submodules of $M$.

It is well-known that for any $M_R, \text{Sat}_\tau(M)$ is an upper continuous modular lattice with respect to the inclusion $\subseteq$ and the operations $\lor$ and $\land$ defined as follows:

$$\lor \{N_i\}_{i \in I} := \sum \{N_i\}_{i \in I} \quad \text{and} \quad \land \{N_i\}_{i \in I} := \bigcap \{N_i\}_{i \in I},$$

having least element $\tau(M)$ and greatest element $M$ (see [8, Chapter 9, Proposition 4.1]).

The reader is referred to [8] for more about hereditary torsion theories.

1. Linear morphisms of lattices and lattice preradicals

In this section we recall from [3] and [4] the concepts of a linear morphism and of a lattice preradical, respectively, and list some of their basic properties. We also present from [5] the concept of a weakly lattice preradical.

As in [3], a mapping $f : L \to L'$ between a lattice $L$ with least element 0 and greatest element 1 and a lattice $L'$ with least element $0'$ and greatest element $1'$ is called a linear morphism if there exist $k \in L$, called a kernel of $f$, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \lor k)$, $\forall x \in L$.
- $f$ induces a lattice isomorphism $	ilde{f} : 1/k \sim a'/0'$, $\tilde{f}(x) = f(x)$, $\forall x \in 1/k$.

If $f : L \to L'$ is a linear morphism of lattices, then $f$ is an increasing mapping, commutes with arbitrary joins (i.e., $f(\lor \{x_i\}_{i \in I}) = \lor \{f(x_i)\}_{i \in I}$ for any family $(x_i)_{i \in I}$ of elements of $L$, provided both joins exist), preserves intervals (i.e., for any $u \leq v$ in $L$, one has $f(v/u) = f(v)/f(u)$), and its kernel $k$ is uniquely determined.

As in [3], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, (for “linear modular”) if for any $L, L' \in \mathcal{M}$ one takes as morphisms from $L$ to $L'$ all the linear morphisms from $L$ to $L'$.

The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices. The monomorphisms (respectively, epimorphisms) in the category $\mathcal{LM}$ are exactly the injective (respectively, surjective) linear morphisms. Moreover, the subobjects of $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ for any $a \in L$.

As in [5], a non-empty class $\mathcal{C}$ of lattices is said to be weakly hereditary if $a/0 \in \mathcal{C}$ for any $L \in \mathcal{C}$ and $a \in L$. According to [6], an abstract class of lattices is a subclass $\emptyset \neq \mathcal{C} \subseteq \mathcal{L}$ which is closed under lattice isomorphisms, i.e., if $L, K \in \mathcal{L}$, $K \cong L$, and $L \in \mathcal{C}$, then $K \in \mathcal{C}$. Thus, a hereditary class of lattices as defined in [6] is nothing else than a weakly hereditary class which additionally is an abstract class.
For any non-empty subclass \( C \) of \( \mathcal{M} \) we shall denote by \( \mathcal{LC} \) the full subcategory of \( \mathcal{LM} \) having \( C \) as the class of its objects.

Let \( C \) be a weakly hereditary subclass of \( \mathcal{M} \). As in [5], a weakly lattice preradical on \( C \) is any functor \( r : \mathcal{LC} \rightarrow \mathcal{LC} \) satisfying the following two conditions.

- \( r(L) \) is an initial interval of \( L \) for any \( L \in \mathcal{LC} \).
- For any morphism \( f : L \rightarrow L' \) in \( \mathcal{LC} \), \( r(f) : r(L) \rightarrow r(L') \) is the restriction and corestriction of \( f \) to \( r(L) \) and \( r(L') \), respectively.

The lattice preradicals defined in [4] are precisely the weakly lattice preradicals on hereditary classes \( C \subseteq \mathcal{M} \). As in the case of “true” lattice preradicals, for a weakly lattice preradical \( r \) on the weakly hereditary class \( C \subseteq \mathcal{M} \), we set \( r(a/0) := a r/0 \) for any \( a \in L \) and \( L \in \mathcal{C} \).

If \( a \leq b \) in \( L \) then \( a/0, b/0 \) are both in \( C \) because \( C \) is weakly hereditary. The inclusion mapping \( \iota : a/0 \hookrightarrow b/0 \) is clearly a linear morphism, thus it is a morphism in \( \mathcal{LC} \). Applying now \( r \) we obtain \( r(\iota) : a r/0 \rightarrow b r/0 \) as a restriction of \( \iota \), and so \( a r \leq b r \).

2. Connections between lattice preradicals and module preradicals

This section contains the main results of the paper. We first show that any lattice preradical naturally induces a module preradical, or more generally a preradical on any locally small Abelian category, but not conversely. Then, we introduce the concept of a linearly closed subcategory of \( \mathcal{LM} \) and show that, based on this, the main results of [5] about lattice preradicals on \( C_{11} \) lattices also hold for preradicals on linearly closed subcategories that are weakly hereditary; so, they can be at once applied to Grothendieck categories and module categories equipped with hereditary torsion theories.

**Proposition 2.1.** For any lattice preradical \( r \) on \( \mathcal{LM} \), the assignment \( M_{R} \mapsto M^{r} \) defines a preradical \( \mathcal{r} \) on \( \text{Mod}-R \).

**Proof.** Recall that, when we specialize the notation \( a r/0 : r(a/0) \), \( a \in L \), \( L \in \mathcal{LM} \), for \( L = \mathcal{L}(M_{R}) \) and \( a = M \), we have \( M^{r}/0 = r(\mathcal{L}(M_{R})) \) in the lattice \( \mathcal{L}(M_{R}) = M/0 \).

Clearly \( \mathcal{r}(M) := M^{r} \leq M \). Let \( f : M \rightarrow M' \) be a morphism of right \( R \)-modules. Then \( f \) induces a mapping

\[
\mathcal{f} : \mathcal{L}(M) \rightarrow \mathcal{L}(M'), \quad \mathcal{f}(N) = f(N), \quad \forall N \leq M,
\]

which is a linear morphism of lattices. Since \( r \) is a preradical on \( \mathcal{LM} \), we have

\[
\mathcal{f}(M^{r}/0) = \mathcal{f}(r(\mathcal{L}(M))) \leq r(\mathcal{L}(M')) = M'^{r}/0,
\]

and so, \( \mathcal{f}(M^{r}) \subseteq M'^{r} \), that is, \( f(\mathcal{r}(M)) \subseteq \mathcal{r}(M') \). Thus \( \mathcal{r} \) is a module preradical. \( \Box \)
More generally, we may consider instead of Mod-$R$ any locally small Abelian category. Recall that an Abelian category $\mathcal{A}$ is said to be \textit{locally small} if the class $\mathcal{L}(X)$ of all subobjects of each object $X$ of $\mathcal{A}$ is a set, and in this case, $\mathcal{L}(X)$ is actually a modular lattice. We shall use the standard notation $A \subseteq X$ to designate an element $A \in \mathcal{L}(X)$. As it is well-known, any Grothendieck category is locally small. To extend Proposition 2.1 to a locally small Abelian category $\mathcal{A}$, it suffices to observe that, by [4, Lemma 5.1], for any morphism $f : X \to Y$ in $\mathcal{A}$, the induced mapping $f^* : \mathcal{L}(X) \to \mathcal{L}(Y)$, $f^*(A) = f(A)$, $\forall A \subseteq X$, is a linear morphism of lattices.

The next example shows that a module preradical does not necessarily define a lattice preradical.

**Example 2.2.** For any $M \in \mathbb{Z}$-$\text{Mod}$, denote $r(M) = \{x \in M \mid 2x = 0\}$. Then $r$ is a preradical on $\mathbb{Z}$-$\text{Mod}$. We claim that there is no lattice preradical $r$ such that $r$ is obtained from $r$ as in Proposition 2.1.

To see this, suppose that such an $r$ exists. Consider the cyclic Abelian groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Since their lattices of subgroups $\mathcal{L}(\mathbb{Z}_2)$ and $\mathcal{L}(\mathbb{Z}_3)$ are two-element chains, they are isomorphic, and let $\varphi : \mathcal{L}(\mathbb{Z}_2) \xrightarrow{\sim} \mathcal{L}(\mathbb{Z}_3)$ be the (unique) lattice isomorphism. Then $\varphi(r(\mathcal{L}(\mathbb{Z}_2))) \subseteq r(\mathcal{L}(\mathbb{Z}_3))$. But

$$\mathbb{Z}_2^r = r(\mathbb{Z}_2) = \mathbb{Z}_2 \quad \text{and} \quad \mathbb{Z}_3^r = r(\mathbb{Z}_3) = 0,$$

so $r(\mathcal{L}(\mathbb{Z}_2)) = \{\mathbb{Z}_2, 0\}$ and $r(\mathcal{L}(\mathbb{Z}_3)) = \{0\}$, and then

$$\mathbb{Z}_3 = \varphi(\mathbb{Z}_2) \in \varphi(r(\mathcal{L}(\mathbb{Z}_2))) = \{0\},$$

which is a contradiction. \qed

We are now going to investigate when a module preradical produces a sort of a lattice preradical. Thus, we introduce the concept of a \textit{linearly closed} subcategory of the category $\mathcal{LM}$; these are subcategories of $\mathcal{LM}$ that are not necessarily full but enjoy some natural conditions that are in particular satisfied when considering subcategories of locally small Abelian categories or subcategories associated with $\tau$-saturated submodules with respect to a hereditary torsion theory $\tau$ on the category Mod-$R$.

**Definition 2.3.** Let $\mathcal{SC}$ be a subcategory (not necessarily full) of $\mathcal{LM}$ having as class of objects a non-empty subclass $\mathcal{C}$ of $\mathcal{M}$. We say that $\mathcal{SC}$ is \textit{linearly closed} if its class of morphisms $\text{Mor}(\mathcal{SC})$ satisfies the following four properties.

1. If $L \in \mathcal{C}$, $a \in L$, and $a/0 \in \mathcal{C}$, then the inclusion mapping

$$i : a/0 \hookrightarrow L, \quad i(x) = x, \forall x \in a/0,$$
(2) If $L \in C$, $a \in L$, and $1/a \in C$, then the linear morphism

$$p : L \rightarrow 1/a, \quad p(x) = x \lor a, \forall x \in L,$$

is in $\text{Mor}(SC)$.

(3) If $f : L \rightarrow L'$ is in $\text{Mor}(SC)$, $k$ is the kernel of $f$, and $a' \in L'$ is such that $\bar{f} : 1/k \sim \rightarrow a'/0'$ is the induced isomorphism, then

$$1/k \in C, \quad a'/0' \in C, \quad \text{and} \quad \bar{f} \in \text{Mor}(SC).$$

(4) If $f : L \sim \rightarrow L'$ is in $\text{Mor}(SC)$ and is an isomorphism in $\mathcal{LM}$, then its inverse $f^{-1}$ is in $\text{Mor}(SC)$ (i.e., $f$ is an isomorphism in $SC$). □

The next result has a series of consequences that will be essentially used in our forthcoming paper [5] investigating the behavior under lattice pre-radicals of the condition $(C_{11})$ in modular lattices.

**Proposition 2.4.** Let $SC$ be a linearly closed subcategory of $\mathcal{LM}$, let $f : L \rightarrow L'$ be a morphism in $SC$ with kernel $k$, and let $a, b \in L$ such that $a/0$ and $1'/f(b)$ are in $SC$. Then $a/((b \lor k) \land a)$ and $f(a \lor b)/f(b)$ are both in $SC$, and the canonical morphism

$$\bar{g} : a/((b \lor k) \land a) \rightarrow f(a \lor b)/f(b), \quad x \mapsto f(x) \lor f(b),$$

induced by $f$ is an isomorphism in $\text{Mor}(SC)$.

**Proof.** By Definition 2.3(1), the inclusion mapping $i : a/0 \hookrightarrow L$ is in $\text{Mor}(SC)$, and, by Definition 2.3(2) the projection

$$p : L' \rightarrow 1'/f(b), \quad p(y) = y \lor f(b), \quad \forall y \in L',$$

is also in $\text{Mor}(SC)$. Thus $g := p \circ f \circ i : a/0 \rightarrow 1'/f(b)$ is in $\text{Mor}(SC)$.

The kernel of $g$ is $(b \lor k) \land a$. Indeed, for $x \in a/0$, we have

$$g(x) = f(b) \iff f(x) \lor f(b) = f(b) \iff f(x \lor b) = f(b) \iff x \lor b \lor k = b \lor k \iff x \leq b \lor k \iff x \leq (b \lor k) \land a.$$

Since $g(a) = f(a \lor b)$, it follows that the isomorphism induced by the linear morphism $g$ is

$$\bar{g} : a/((b \lor k) \land a) \rightarrow f(a \lor b)/f(b), \quad x \mapsto f(x) \lor f(b).$$

By Definition 2.3(3), we obtain the desired conclusion. □
Corollary 2.5. The following assertions hold for a linearly closed subcategory $\mathcal{SC}$ of $\mathcal{LM}$, a morphism $f : L \to L'$ in $\mathcal{SC}$ with kernel $k$, and elements $a, b \in L$.

(1) If $a/0 \in \mathcal{SC}$, then both intervals $a/(a \wedge k)$ and $f(a)/0'$ are in $\mathcal{SC}$, and the canonical morphism
\[ \alpha : a/(a \wedge k) \to f(a)/0' \]
induced by $f$ is an isomorphism in $\text{Mor}(\mathcal{SC})$.

(2) If $1'/f(b) \in \mathcal{SC}$, then both intervals $1/(b \vee k)$ and $f(1)/f(b)$ are in $\mathcal{SC}$, and the canonical morphism
\[ \beta : 1/(b \vee k) \to f(1)/f(b) \]
induced by $f$ is an isomorphism in $\text{Mor}(\mathcal{SC})$.

Proof. (1) Apply Proposition 2.4 first for $b = 0$, and then for $a = 1$. □

Corollary 2.6. The following assertions hold for a linearly closed subcategory $\mathcal{SC}$ of $\mathcal{LM}$, $L \in \mathcal{C}$, and $a, b \in L$.

(1) If $a/0 \in \mathcal{C}$ and $1/b \in \mathcal{C}$, then $a/(a \wedge b) \in \mathcal{C}$, $(a \vee b)/b \in \mathcal{C}$, and the canonical isomorphisms
\[ \varphi : a/(a \wedge b) \simeq (a \vee b)/b, \quad \varphi(x) = x \vee b, \quad \forall x \in a/(a \wedge b), \]
\[ \psi : (a \vee b)/b \simeq a/(a \wedge b), \quad \psi(y) = y \wedge a, \quad \forall y \in (a \vee b)/b, \]
are both in $\text{Mor}(\mathcal{SC})$.

(2) Suppose that $1 = a \vee b$ (this means that $1 = a \vee b$ and $a \wedge b = 0$). If $a/0 \in \mathcal{C}$ and $1/b \in \mathcal{C}$, then the linear morphism
\[ q : L \to a/0, \quad q(x) := (x \vee b) \wedge a, \quad \forall x \in L, \]
is in $\text{Mor}(\mathcal{SC})$. Moreover, $q$ is a surjective linear morphism with kernel $b$.

(3) If $0/0 \in \mathcal{C}$, then, the mapping $o : L \to 0/0, \quad o(x) = 0, \quad \forall x \in L$, is in $\text{Mor}(\mathcal{SC})$.

(4) If $K \in \mathcal{C}$, $0/0 \in \mathcal{C}$, and there exists a morphism from $K$ to $L$ in $\text{Mor}(\mathcal{SC})$, then the mapping $K \to L, \ x \mapsto 0$, is in $\text{Mor}(\mathcal{SC})$.

Proof. (1) Apply Proposition 2.4 for $L' = L$ and $f = 1_L$. Then $\overline{g} = \varphi$ is in $\text{Mor}(\mathcal{SC})$. Since $\psi = \varphi^{-1}$, by Definition 2.3(4), we have $\psi \in \text{Mor}(\mathcal{SC})$.

(2) With notation from (1) above, we have $q = \psi \circ p$, where
\[ p : L \to 1/b, \quad p(x) = x \lor b, \quad \forall x \in L. \]

For the last part of (2), see [4, Example 0.2(3)].

(3) Take \( a = 0 \) and \( b = 1 \) in (2).

(4) Compose the inclusion mapping of \( 0/0 \) into \( L \) with the previous mapping \( o \) and the supposed morphism from \( K \) to \( L \). \( \square \)

We present now two examples where linearly closed subcategories naturally occur: in locally small Abelian categories and in \( \tau \)-saturated submodules with respect to a hereditary torsion theory \( \tau \) on the category Mod-\( R \).

**Example 2.7.** Let \( \mathcal{X} \) be a non-empty class of objects of a locally small Abelian category \( \mathcal{A} \), in particular a non-empty class of right \( R \)-modules. We assume that \( \mathcal{X} \) is hereditary, i.e., it is closed under subobjects; this means that for every \( X \in \mathcal{X} \) and subobject \( Y \) of \( X \) in \( \mathcal{A} \), we have \( Y \in \mathcal{X} \).

For any \( X' \subseteq X \) in \( \mathcal{A} \), we denote by \([X', X]\) the interval in the lattice \( \mathcal{L}(X) \), and by

\[ \varphi_{X/X'} : [X', X] \xrightarrow{\sim} \mathcal{L}(X/X') \]

the canonical lattice isomorphism \( Z \mapsto Z/X' \), which is clearly a linear morphism of lattices.

We shall associate to \( \mathcal{X} \) a linearly closed subcategory \( \mathcal{SC}_\mathcal{X} \) having

\[ \mathcal{C}_\mathcal{X} := \{ [X', X] \mid X \in \mathcal{X}, \ X' \subseteq X \} \]

as class of objects, and as morphisms those mappings that are induced by morphisms \( f : X/X' \to Y/Y' \) in \( \mathcal{A} \), i.e., arise as compositions

\[ [X', X] \xrightarrow{\varphi_{X/X'}} \mathcal{L}(X/X') \xrightarrow{f_*} \mathcal{L}(Y/Y') \xrightarrow{\varphi_{Y/Y'}^{-1}} [Y', Y]. \]

Recall that for any morphism \( f : A \to B \) in \( \mathcal{A} \) we denoted by \( f_* \) the so called direct image mapping

\[ f_* : \mathcal{L}(A) \to \mathcal{L}(B), \quad f_*(A') = f(A'), \ \forall A' \in \mathcal{L}(A). \]

By [4, Lemma 5.1], any such mapping \( f_* \) is a linear morphism of lattices, so, the morphisms in \( \mathcal{SC}_\mathcal{X} \), as compositions of linear morphisms of lattices, are also linear morphisms of lattices.

Notice that the transition from morphisms in \( \mathcal{A} \) to their direct image mappings is functorial, i.e., for any morphisms \( A \xrightarrow{f} B \) and \( B \xrightarrow{g} C \) we have \( (g \circ f)_* = g_* \circ f_* \) and \( (1_A)_* = 1_{\mathcal{L}(A)} \). Therefore, if \( f \) is an isomorphism in \( \mathcal{A} \), then \( f_* \) is a linear lattice isomorphism and \( (f_*)^{-1} = (f^{-1})_* \).

Clearly, \( \mathcal{SC}_\mathcal{X} \) is a subcategory, not necessarily full, of the category \( \mathcal{LM} \). We are now going to show that \( \mathcal{SC}_\mathcal{X} \) is indeed a linearly closed subcategory of \( \mathcal{LM} \), i.e., it verifies the properties (1) - (4) of Definition 2.3.
For property (1), let \([X', X] \in \mathcal{C}_X\), and let \(Y \in [X', X]\). Because the class \(\mathcal{X}\) is hereditary, we have \(Y \in \mathcal{X}\). Clearly, the inclusion mapping \([X', Y] \hookrightarrow [X', X]\) is induced by the inclusion morphism \(Y/X' \hookrightarrow X/X'\) in \(\mathcal{A}\), so \(t \in \text{Mor}(\mathcal{SC}_X)\), as desired.

For property (2), let \([X', X] \in \mathcal{C}_X\), and let \(Y \in [X', X]\). Then \(Y \in \mathcal{X}\). We have to prove that the mapping
\[
\pi : [X', X] \rightarrow [Y, X], \quad \pi(Z) = Y + Z, \quad \forall Z \in [X', X],
\]
is induced by a certain morphism in \(\mathcal{A}\), namely by the canonical epimorphism \(q : X/X' \twoheadrightarrow X/Y\) in \(\mathcal{A}\), i.e.,
\[
\pi = \varphi_{X/Y}^{-1} \circ q_* \circ \varphi_{X/X'}.
\]
Indeed
\[
(\varphi_{X/Y}^{-1} \circ q_* \circ \varphi_{X/X'})(Z) = (\varphi_{X/Y}^{-1} \circ q_*)(Z/X') = \varphi_{X/Y}^{-1}((Y + Z)/Y) = Y + Z = \pi(Z), \quad \forall Z \in [X', X].
\]

To verify the property (3), let \(\alpha : [X', X] \rightarrow [Y', Y]\) be a morphism in \(\mathcal{SC}_X\). This means that \(\alpha\) is induced by a morphism \(f : X/X' \rightarrow Y/Y'\) in \(\mathcal{A}\), i.e.,
\[
\alpha = \varphi_{Y/Y'}^{-1} \circ f_* \circ \varphi_{X/X'}.
\]
Set \(K := \text{Ker}(f)\) and \(I := \text{Im}(f)\). Since \(\mathcal{A}\) is an Abelian category, we have \(K = U/X'\) and \(I = V/Y'\) for some \(X' \subseteq U \subseteq X\) and \(Y' \subseteq V \subseteq Y\). Now, observe that \(V \in \mathcal{C}\) because the given class \(\mathcal{C}\) is hereditary, so \([U, X] \in \mathcal{C}_X\) and \([Y, V] \in \mathcal{C}_X\).

Further, let
\[
\overline{f} : (X/X')/(U/X') \rightarrow V/Y' \quad \text{and} \quad h : X/U \rightarrow (X/X')/(U/X')
\]
be the canonical isomorphisms in \(\mathcal{A}\), and set \(g := \overline{f} \circ h\). Then \(g_* = \overline{f}_* \circ h_*\) is an isomorphism in \(\mathcal{LM}\). If we set \(\overline{\alpha} := \varphi_{V/Y'} \circ g_* \circ \varphi_{X/U}\), then it is easily checked that the obtained isomorphism \(\overline{\alpha} : [U, X] \rightarrow [Y', V]\) in \(\mathcal{LM}\) is a restriction of the given morphism \(\alpha\), i.e., \(\overline{\alpha}(Z) = \alpha(Z), \quad \forall Z \in [U, X]\). To conclude that \(\overline{\alpha} \in \text{Mor}(\mathcal{SC}_X)\), we have to prove that \(U\) is the kernel of the given linear mapping \(\alpha\), i.e., \(\alpha(W + U) = \alpha(W), \quad \forall W \in [X', X]\).

Indeed, for any \(W \in [X', X]\), we have
\[
\alpha(W + U) = (\varphi_{Y/Y'}^{-1} \circ f_* \circ \varphi_{X/X'})(W + U) = (\varphi_{Y/Y'}^{-1} \circ f_*)((W + U)/X') = \\
= \varphi_{Y/Y'}^{-1}((f(W + U)/X')) = \varphi_{Y/Y'}^{-1}(f(W/X') + f(U/X')) = \\
= \varphi_{Y/Y'}^{-1}(f(W/X') + f(K)) = \varphi_{Y/Y'}^{-1}(f(W/X')) = \alpha(W).
\]
To prove the property (4), let \( \alpha : [X', X] \to [Y', Y] \), \( \alpha \in \text{Mor} (\mathcal{SC}_X) \). This means that \( \alpha \) is induced by a morphism \( f : X/X' \to Y/Y' \) in \( \mathcal{A} \), i.e.,
\[
\alpha = \varphi_{Y'/Y}^{-1} \circ f_* \circ \varphi_{X/X'}^{-1}.
\]
Assume that \( \alpha \) is an isomorphism in \( \mathcal{LM} \), so a bijective mapping. Then \( f_* \) is also a bijective mapping.

Let \( K := \text{Ker} (f) \). Then, \( f_* (K) = f(K) = 0 = f_*(0) \), where 0 is the zero object of \( \mathcal{A} \), so \( K = 0 \) because \( f_* \) is an injective mapping. Thus \( f \) is a monomorphism. We also have \( \alpha(X) = Y \) because \( \alpha \), as a lattice isomorphism, carries the greatest element of \( [X', X] \) onto the greatest element of \( [Y', Y] \). Then \( f(X/X') = Y/Y' \), i.e., \( f \) is an epimorphism, so a bimorphism. Thus \( f \) is an isomorphism in \( \mathcal{A} \). This implies that \( \alpha^{-1} \) is induced by \( f^{-1} \), i.e., \( \alpha^{-1} \) is an isomorphism in \( \text{Mor} (\mathcal{SC}_X) \), as desired. \( \square \)

We shall discuss now another circumstance where the linearly closed subcategories naturally occur, namely in lattices of \( \tau \)-saturated submodules with respect to a hereditary torsion theory \( \tau \) on the category \( \text{Mod-} R \). To do that, we recall the following result.

**Lemma 2.8.** ([1, Lemma 3.4.4]). The following statements hold for a module \( M_R \) and submodules \( P \subseteq N \) of \( M_R \).

1. The mapping \( \alpha : \text{Sat}_\tau (N/P) \to \text{Sat}_\tau (N/P) \), \( X/P \mapsto X/P \), is a lattice isomorphism.
2. \( \text{Sat}_\tau (N) \simeq \text{Sat}_\tau (N) \).
3. If \( M/N \in \mathcal{T} \), then \( \text{Sat}_\tau (M) \simeq \text{Sat}_\tau (N) \).
4. If \( N, P \in \text{Sat}_\tau (M) \), then the assignment \( X \mapsto X/P \) defines a lattice isomorphism from the interval \( [P, N] \) of the lattice \( \text{Sat}_\tau (M) \) onto the lattice \( \text{Sat}_\tau (N/P) \). \( \square \)

**Example 2.9.** Let \( \tau = (\mathcal{T}, \mathcal{F}) \) be a hereditary torsion theory on \( \text{Mod-} R \), and let \( \mathcal{H} \) be a non-empty class of right \( R \)-modules which is \( \tau \)-hereditary. Recall from [5] that \( \mathcal{H} \) is said to be \( \tau \)-hereditary if for any \( M \in \mathcal{H} \) and \( N \in \text{Sat}_\tau (M) \) one has \( N \in \mathcal{H} \).

For any \( M \in \mathcal{H} \) and \( M' \in \text{Sat}_\tau (M) \), we denote by \( [M', M] \) the interval in the lattice \( \text{Sat}_\tau (M) \), and by
\[
\psi_{M/M'} : [M', M] \cong \text{Sat}_\tau (M/M'), \psi(N) := N/M', \forall N \in [M', M],
\]
the canonical lattice isomorphism in Lemma 2.8(4), which is clearly a linear morphism of lattices.

We shall associate to \( \mathcal{H} \) a linearly closed subcategory \( \mathcal{SC}_\mathcal{H} \) having
\[
\mathcal{C}_\mathcal{H} := \{ [M', M] \mid M \in \mathcal{H}, M' \in \text{Sat}_\tau (M) \}.
\]
as class of objects and as morphisms those mappings that are induced by morphisms $f : M/M' \to P/P'$ in Mod-$R$, i.e., arise as compositions

$$[M', M] \xrightarrow{\psi_{M/M'}} \text{Sat}_\tau(M/M') \xrightarrow{f_\tau} \text{Sat}_\tau(P/P') \xrightarrow{\psi_{P/P'}^{-1}} [P', P].$$

where, for any morphism $f : A \to B$ in Mod-$R$, $f_\tau$ denotes the mapping

$$f_\tau : \text{Sat}_\tau(A) \to \text{Sat}_\tau(B), f_\tau(X) = \overline{f(X)}, \forall X \in \text{Sat}_\tau(A).$$

Notice that $f_\tau$ is a linear morphism of lattices by [4, Lemma 6.6]. We deduce that the morphisms in $\mathcal{SC}_H$, as compositions of linear morphisms of lattices, are also so.

We are now going to show that $\mathcal{SC}_H$ is indeed a linearly closed subcategory of $\mathcal{LM}$, i.e., it verifies the properties (1) - (4) of Definition 2.3. Essentially, we shall proceed as in Example 2.7 by replacing the lattices $\mathcal{L}(X/X')$ with the lattices $\text{Sat}_\tau(M/M')$, and the intervals $[X', X]$ in the lattice $\mathcal{L}(X)$ with the intervals $[M', M]$ in the lattice $\text{Sat}_\tau(M)$.

For instance, to check the property (1), let $[M', M] \in C_H$ and $N \in [M', M]$. Because the class $H$ is $\tau$-hereditary, we have $N \in H$. Clearly, the inclusion mapping $\iota : [M', N] \to [M', M]$ is induced by the inclusion morphism $N/M' \hookrightarrow M/M'$ in Mod-$R$, so $\iota \in \text{Mor}(\mathcal{SC}_H)$, as desired.

Similarly, to prove the property (2), let $[M', M] \in C_H$ and $N \in [M', M]$. Then $N \in H$. We have to prove that the mapping

$$\pi : [M', M] \to [N, M], \pi(P) = N \lor P, \forall P \in [M', M],$$

is induced by a certain morphism in Mod-$R$, namely by the canonical epimorphism $q : M/M' \to M/N$, $q(U/M') = (N + U)/N$, in Mod-$R$, i.e.,

$$\pi = \psi_{M/N}^{-1} \circ q_\tau \circ \psi_{M/M'}.$$  

Indeed, $q_\tau(P/M') = \overline{q(P/M')} = (N + P)/N = (N + P)/N = (N \lor P)/N$, so we have

$$(\psi_{M/N}^{-1} \circ q_\tau \circ \psi_{M/M'})(P) = (\psi_{M/N}^{-1} \circ q_\tau)(P/M') = \psi_{M/N}(N \lor P) = N \lor P = \pi(P), \forall P \in [M', M].$$

To verify the property (3), let $\alpha : [M', M] \to [N', N]$ be a morphism in $\mathcal{SC}_H$. This means that $\alpha$ is induced by a morphism $f : M/M' \to N/N'$ in Mod-$R$, i.e.,

$$\alpha = \psi_{N/N'}^{-1} \circ f_\tau \circ \psi_{M/M'}.$$  

Set $K := \text{Ker}(f)$ and $I := \text{Im}(f)$. We have $K = U/M'$ and $I = V/N'$ for some $M' \leq U \leq M$ and $N' \leq V \leq N$.

Further, let
We have \( U, M \) = 0 = \( h : M/U \xrightarrow{\sim} (M/M')/(U/M') \) be the canonical module isomorphisms, and set \( g := f \circ h \). Then \( g_\tau = f_\tau \circ h_\tau \) is an isomorphism in \( \mathcal{LM} \).

Because \( \overline{U} \in \text{Sat}_\tau(M) \), \( \overline{V} \in \text{Sat}_\tau(N) \), and the class \( \mathcal{H} \) is hereditary, we have \([\overline{U}, M], [N', \overline{V}] \in \mathcal{C}_\mathcal{H}\). We are going to prove that there exists a linear lattice isomorphism \( \beta : [\overline{U}, M] \xrightarrow{\sim} [N', \overline{V}] \) such that \( \beta \) is the restriction of the given morphism \( \alpha \).

Indeed, the lattice isomorphism \( g_\tau : \text{Sat}_\tau(M/U) \xrightarrow{\sim} \text{Sat}_\tau(V/N') \) yields by Lemma 2.8 the following sequence of canonical lattice isomorphisms

\[
[\overline{U}, M] \xrightarrow{\sim} \text{Sat}_\tau(M/U) \xrightarrow{\sim} \text{Sat}_\tau(\overline{V}/N') \xrightarrow{\sim} [N', \overline{V}].
\]

It is straightforward to check that their composition \( \beta \) is exactly the restriction of the given morphism \( \alpha : [M', M] \longrightarrow [N', N] \) in \( \mathcal{SC}_\mathcal{H} \), i.e., \( \alpha(Z) = \beta(Z), \forall Z \in [U, M] \).

To conclude, we have to prove that \( \overline{U} \) is the kernel of the given linear mapping \( \alpha \), i.e.,

\[
\alpha(W \vee \overline{U}) = \alpha(W), \forall W \in [M', M].
\]

First, notice that \( f(K) \subseteq f(\overline{K}) \) (see the proof of [4, Lemma 6.6]), so \( \overline{U} \subseteq f(\overline{K}) \subseteq f(K) = \overline{\partial} \), and then \( f(K) = f(\overline{K}) = \overline{\partial} \). We have

\[
\alpha(W \vee \overline{U}) = (\psi_{N'/N}^{-1} \circ f_\tau \circ \psi_{M'/M})(W \vee \overline{U}) = (\psi_{N'/N}^{-1} \circ f_\tau)((W \vee \overline{U})/M') =
\]

\[
= \psi_{N'/N}^{-1}(f_\tau((W \vee \overline{U})/M')) = \psi_{N'/N}^{-1}(f_\tau(\psi_{M'/M}(W))) =
\]

\[
= (\psi_{N'/N}^{-1} \circ f_\tau \circ \psi_{M'/M})(W) = \alpha(W),
\]

as desired, because

\[
f_\tau((W \vee \overline{U})/M') = f_\tau((W + \overline{U})/M') = f_\tau((W + \overline{U})/M') =
\]

\[
= f_\tau((W + U)/M') = f_\tau((W/M') + (U/M')) = f_\tau((W/M') \vee (U/M')) =
\]

\[
f_\tau(W/M') \vee f_\tau(\overline{K}) = f(W/M') \vee f(\overline{K}) = f(W/M') \vee \overline{\partial} = f_\tau(\psi_{M'/M}(W)).
\]

To prove the property (4), let \( \alpha : [M', M] \longrightarrow [N', N], \alpha \in \text{Mor}(\mathcal{SC}_\mathcal{H}) \). This means that \( \alpha \) is induced by a morphism \( f : M/M' \longrightarrow N/N' \) in \( \text{Mod}-R \), i.e.,

\[
\alpha = \psi_{N'/N}^{-1} \circ f_\tau \circ \psi_{M'/M'}.
\]

Assume that \( \alpha \) is an isomorphism in \( \mathcal{LM} \), so a bijective mapping. Then \( f_\tau \) is also a bijective mapping. Notice that \( M/M', N/N' \in \mathcal{F} \) because \( M' \in \text{Sat}_\tau(M) \) and \( N' \in \text{Sat}_\tau(N) \).

Let \( K := \text{Ker}(f) \). Then, \( f_\tau(K) = f(\overline{K}) = \overline{\partial} = 0 = f_\tau(0) \), so \( K = 0 \) because \( f_\tau \) is an injective mapping, so \( f \) is a monomorphism.

We have also \( \alpha(M) = N \) because \( \alpha \), as a lattice isomorphism, carries the greatest element of \([M', M]\) onto the greatest element of \([N', N]\). Then \( f(M/M') = N/N' \), i.e., \( f \) is an epimorphism, so an isomorphism in \( \text{Mod}-R \). This implies that \( \alpha^{-1} \) is induced by \( f^{-1} \), which shows that \( \alpha^{-1} \) is an isomorphism in \( \text{Mor}(\mathcal{SC}_\mathcal{H}) \), as desired. \( \square \)
3. Preradicals on linearly closed subcategories of $\mathcal{LM}$

In this section we define the more general concept of a preradical on a linearly closed subcategory of $\mathcal{LM}$ and show that we can associate to preradicals on locally small Abelian categories and module categories equipped with hereditary torsion theories lattice preradicals on the linearly closed subcategories $\mathcal{SC}_X$ and $\mathcal{SC}_H$ discussed in Examples 2.7 and 2.9, respectively. Finally we show that how the main results of [5] also hold for any preradical on a linearly closed subcategory of $\mathcal{LM}$ which is weakly hereditary.

**Proposition 3.1.** The following assertions are equivalent for a a linearly closed subcategory $\mathcal{SC}$ of $\mathcal{LM}$.

1. $\mathcal{C}$ is weakly hereditary.
2. The monomorphisms in the category $\mathcal{SC}$ are injective.
3. For any $L \in \mathcal{C}$, the subobjects of $L$ in the category $\mathcal{SC}$ can be regarded as the initial intervals $a/0$ of $L = 1/0$, $a \in L$.

**Proof.** (1)$\implies$(2): Let $f : L \to L'$ be a monomorphism in $\mathcal{SC}$. If $k$ is the kernel of $f$, then $K := k/0 \in \mathcal{C}$ since $\mathcal{C}$ is weakly hereditary. By Definition 2.3, the inclusion mapping $\kappa : K \hookrightarrow L$ is in $\text{Mor}(\mathcal{SC})$. Also, since $\mathcal{C}$ is weakly hereditary, we have $0/0 \in \mathcal{C}$, and by Corollary 2.6(4) the zero mapping $o : K \to L$ is in $\text{Mor}(\mathcal{SC})$. We have $f \circ \kappa = f \circ o$, and since $f$ is a monomorphism, we deduce that $\kappa = o$, thus $k = 0$, and consequently, $f$ is injective.

(2)$\implies$(3): Let $(S, \alpha)$ be a subobject of $L$ in $\mathcal{SC}$. Then $\alpha$ is a monomorphism, thus injective by (2). By Definition 2.3, its image $a/0 \in \mathcal{C}$, for $a \in L$, and since its kernel is zero, $\alpha$ induces an isomorphism $\overline{\alpha} : S \xrightarrow{\sim} a/0$, which is in $\text{Mor}(\mathcal{SC})$. Since the inclusion mapping of $i : a/0 \hookrightarrow L$ is a monomorphism in $\text{Mor}(\mathcal{SC})$, it follows that $(a/0, i)$ is a subobject of $L$ in $\mathcal{SC}$ that is isomorphic to $(S, \alpha)$ via $\overline{\alpha}$.

(3)$\implies$(1): For $a \in L$ and inclusion mapping $i : a/0 \hookrightarrow L$, $(a/0, i)$ is a subobject of $L$ in $\mathcal{SC}$, hence $a/0 \in \mathcal{C}$. \hfill\Box

**Definition 3.2.** Let $\mathcal{SC}$ be a linearly closed subcategory of $\mathcal{LM}$ such that its class of objects $\mathcal{C}$ is weakly hereditary. A lattice preradical on $\mathcal{SC}$ is any functor $r : \mathcal{SC} \to \mathcal{SC}$ satisfying the following two conditions.

1. $r(L) \leq L$, i.e., $r(L)$ is a subobject of $L$, for any $L \in \mathcal{SC}$.
2. For any morphism $f : L \to L'$ in $\mathcal{SC}$, $r(f) : r(L) \to r(L')$ is the restriction and corestriction of $f$ to $r(L)$ and $r(L')$, respectively. \hfill\Box
Let $\mathcal{SC}$ be a linearly closed subcategory of $\mathcal{LM}$ such that its class of objects $\mathcal{C}$ is weakly hereditary, and let $r: \mathcal{SC} \rightarrow \mathcal{SC}$ be a lattice preradical on $\mathcal{SC}$. By Proposition 3.1, for every $L \in \mathcal{C}$ and $a \in L$, the subobject $r(a/0)$ of $L$ in $\mathcal{SC}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$ 

If $a \leq b$ in $L$ then $a/0$, $b/0$ are in $\mathcal{C}$ because $\mathcal{C}$ is weakly hereditary. The inclusion mapping $i : a/0 \hookrightarrow b/0$ is in $\text{Mor}(\mathcal{SC})$ since $\mathcal{SC}$ is linearly closed. Applying $r$ we obtain the morphism $r(i) : a^r/0 \rightarrow b^r/0$ as a restriction of $i$, and so $a^r \subseteq b^r$.

Recall that a preradical on an Abelian category $\mathcal{A}$ is just a subfunctor of the identity functor $1_\mathcal{A}$ of $\mathcal{A}$.

**Proposition 3.3.** Let $\mathcal{X}$ be a hereditary class of objects of a locally small Abelian category $\mathcal{A}$, and let $r$ be a preradical on $\mathcal{A}$. Then $r$ canonically yields a preradical $\varrho$ on the linearly closed subcategory

$$\mathcal{SC}_{\mathcal{X}} := \{ [X', X] \mid X \in \mathcal{X}, X' \subseteq X \}$$

of $\mathcal{LM}$ discussed in Example 2.7.

**Proof.** With notation of Example 2.7, let $[X', X] \in \mathcal{SC}_{\mathcal{X}}$. Then $r(X/X') = Y/X'$ for some $Y \in \mathcal{A}$ with $X' \subseteq Y \subseteq X$. We set $X^r := Y$. Because $\mathcal{X}$ is a hereditary subclass of $\mathcal{A}$, we have $X^r \in \mathcal{X}$, so we can define the following mapping

$$\varrho : \mathcal{SC}_{\mathcal{X}} \rightarrow \mathcal{SC}_{\mathcal{X}}, \varrho([X', X]) := [X', X^r], \forall [X', X] \in \mathcal{SC}_{\mathcal{X}}.$$

By definition, $\varrho([X', X])$ is a subobject of $[X', X]$ for any $[X', X] \in \mathcal{SC}_{\mathcal{X}}$. To conclude that $\varrho$ is a preradical on $\mathcal{SC}_{\mathcal{X}}$, we must show that for any morphism $\alpha : [X', X] \rightarrow [Y', Y]$ in $\mathcal{SC}_{\mathcal{X}}$, we have

$$\alpha(\varrho([X', X])) \subseteq \varrho(\alpha([X', X])), \text{ i.e., } \alpha([X', X^r]) \subseteq [Y', Y^r], \forall [X', X] \in \mathcal{SC}_{\mathcal{X}}.$$

Indeed, by the definition of the morphisms in $\mathcal{SC}_{\mathcal{X}}$, $\alpha$ is induced by a morphism $f : X/X' \rightarrow Y/Y'$ in $\mathcal{A}$, i.e., arises as a composition

$$[X', X] \xrightarrow{\varphi_{X/X'}} \mathcal{L}(X/X') \xrightarrow{f_*} \mathcal{L}(Y/Y') \xrightarrow{\varphi_{Y/Y'}^{-1}} [Y', Y].$$

Now, the morphism $f$ yields a morphism $r(f) : r(X/X') \rightarrow r(Y/Y')$, i.e., a morphism $r(f) : X^r/X' \rightarrow Y^r/Y'$, and then $f_*(r(X/X')) \subseteq Y^r/Y'$. This shows that

$$\alpha([X', X^r]) = (\varphi_{Y/Y'}^{-1} \circ f_* \circ \varphi_{X/X'})([X', X^r]) \subseteq \varphi_{Y/Y'}^{-1}(Y^r/Y') = [Y', Y^r],$$

as desired. $\Box$
Proposition 3.4. Let $\tau = (T, F)$ be a hereditary torsion theory on $\text{Mod-} R$, let $\mathcal{H}$ be a $\tau$-hereditary class of right $R$-modules, and let $r$ be preradical on $\text{Mod-} R$. Then $r$ canonically yields a preradical $\varrho_\tau$ on the linearly closed subcategory

$$\mathcal{S}\mathcal{C}_H := \{ [M', M] | M \in \mathcal{H}, M' \in \text{Sat}_\tau(M) \}$$

of $\mathcal{L}\mathcal{M}$ discussed in Example 2.9.

Proof. With notation of Example 2.9, let $[M', M] \in \mathcal{S}\mathcal{C}_H$. Then $r(M/M') = P/M'$ for some $P \in \text{Mod-} R$ with $M' \leq P \leq M$. We set $M'' := \bar{P}$. Because $M \in \mathcal{H}$ and $\mathcal{H}$ is a $\tau$-hereditary subclass of $\text{Mod-} R$, we have $M'' \in \mathcal{H}$, so we can define the following mapping

$$\varrho_\tau : \mathcal{S}\mathcal{C}_H \rightarrow \mathcal{S}\mathcal{C}_H, \varrho_\tau([M', M]) := [M', M''], \forall [M', M] \in \mathcal{S}\mathcal{C}_H.$$

By definition, $\varrho_\tau([M', M])$ is a subobject of $[M', M]$ for any $[M', M] \in \mathcal{S}\mathcal{C}_H$. To conclude that $\varrho_\tau$ is a preradical on $\mathcal{S}\mathcal{C}_H$, we must show that for any morphism $\alpha : [M', M] \rightarrow [N', N]$ in $\mathcal{S}\mathcal{C}_H$, we have

$$\alpha(\varrho_\tau([M', M])) \subseteq \varrho([N', N]), \text{ i.e., } \alpha([M', M'']) \subseteq [N', N''], \forall [M', M] \in \mathcal{S}\mathcal{C}_H.$$

Indeed, by the definition of the morphisms in $\mathcal{S}\mathcal{C}_H$, $\alpha$ is induced by a morphism $f : M/M' \rightarrow N/N'$ in $\text{Mod-} R$, i.e., arises as a composition

$$[M', M] \xrightarrow{\psi_{M/M'}} \text{Sat}_\tau(M/M') \xrightarrow{f_\tau} \text{Sat}_\tau(N/N') \xrightarrow{\psi_{N/N'}^{-1}} [N', N].$$

Now, the morphism $f$ yields a morphism

$$r(f) : P/M' = r(M/M') \rightarrow r(N/N') = Q/N',$$

i.e., $f(P/M') \subseteq Q/N'$. Then, by [5, Lemma 4.4], $f(\bar{P}/M') \subseteq \bar{Q}/N'$, so

$$f_\tau(M'/M') = f(\bar{P}/M') \subseteq \bar{Q}/N' = \bar{Q}/N' = Q/N' = N'/N'.$$

This shows that

$$\alpha([M', M'']) = (\psi_{N/N'}^{-1} \circ f_\tau \circ \psi_{M/M'})([M', M'']) \subseteq \psi_{N/N'}^{-1}(N'/N') = [N', N'],$$

as desired. \qed

Remarks 3.5. (1) Observe that Proposition 3.3 (respectively, Proposition 3.4) also holds when the given preradical $r$ on the category $\mathcal{A}$ (respectively, $\text{Mod-} R$) is a preradical only on the given hereditary class $\mathcal{X}$ (respectively, $\tau$-hereditary class $\mathcal{H}$) under the additional condition that $\mathcal{X}$ is a cohereditary (respectively, $\mathcal{H}$ is a $\tau$-cohereditary) class. Recall that a non-empty subclass of $\mathcal{A}$ is said to be cohereditary if it is closed under quotient objects, and
if \( \tau = (T, F) \) is a hereditary torsion theory on \( \text{Mod-}R \), then, a non-empty class \( \mathcal{H} \) of right \( R \)-modules is said to be \( \tau \)-cohereditary if for any \( M \in \mathcal{H} \) and \( M' \in \text{Sat}_\tau(M) \) one has \( M/M' \in \mathcal{H} \).

(2) A thorough examination of the proofs in [5] shows that they are performed using only morphisms as in Definition 2.3 and Corollary 2.6. So, all the results of [5], in particular [5, Theorem 2.4] and its Corollary 2.5 also hold for any lattice preradical on a linearly closed subcategory of \( \mathcal{LM} \) which is weakly hereditary.

\[ \square \]

References


Toma Albu
Simion Stoilow Institute of Mathematics of the Romanian Academy
Research Unit 5, P.O. Box 1 - 764
RO - 010145 Bucharest 1, Romania
E-mail: Toma.Albu@imar.ro

Mihai Iosif
Bucharest University, Department of Mathematics
Academiei Str. 14, RO - 010014 Bucharest 1, Romania
E-mail: miosifg@gmail.com