New modulus estimates
in Orlicz-Sobolev classes

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Dedicated to 85 years of Academician C. Andreian Cazacu

Abstract - Under a condition of the Calderon type on \( \varphi \), we show that a homeomorphism \( f \) of finite distortion in \( W^{1, \varphi}_{\text{loc}} \) and, in particular, \( f \in W^{1, p}_{\text{loc}} \) for \( p > n - 1 \) in \( \mathbb{R}^n, n \geq 3 \), is a lower \( Q \)-homeomorphisms with \( Q(x) = [K_I(x, f)]^{\frac{1}{p-1}} \) and a ring \( Q \)-homeomorphism with \( Q(x) = K_I(x, f) \) where \( K_I(x, f) \) is its inner dilatation. Similar statements are valid also for finitely bi-Lipschitz mappings that a far-reaching extension of the well-known classes of isometric and quasiisometric mappings. This makes possible to apply our theories on the local and boundary behavior for the lower and ring \( Q \)-homeomorphisms to all given classes.

Key words and phrases : Sobolev and Orlicz-Sobolev classes, mappings of finite distortion, lower and ring \( Q \)-homeomorphisms, finitely bi-Lipschitz mappings.


1. Introduction

It is well-known that the concept of moduli with weights essentially due to Andreian Cazacu, see, e.g., [1]–[3], see also recent works [4]–[6] of her learner. Here we give new modulus estimates for space mappings that essentially improve the corresponding estimates in our last paper [12]. Since this note is short, we refer readers for definitions, history and explanations either to it or in addition to papers [8]–[11] and [16] and monographs [13] and [14]. Many applications will be published elsewhere.

2. Lower \( Q \)-homeomorphisms and Orlicz–Sobolev classes

Given a mapping \( f : D \to \mathbb{R}^n \) with partial derivatives a.e., recall that \( f'(x) \) denotes the Jacobian matrix of \( f \) at \( x \in D \) if it exists, \( J(x) = J(x, f) = \det f'(x) \) is the Jacobian of \( f \) at \( x \), and \( \|f'(x)\| \) is the operator norm of \( f'(x) \), i.e.,

\[
\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}.
\] (2.1)
We also let
\[ l(f'(x)) = \min \{|f'(x)h| : h \in \mathbb{R}^n, |h| = 1\}. \tag{2.2} \]

The **outer dilatation** of \( f \) at \( x \) is defined by
\[
K_O(x) = K_O(x, f) = \begin{cases} 
\|f'(x)\|^n \frac{1}{|J(x,f)|} & \text{if } J(x,f) \neq 0, \\
1 & \text{if } f'(x) = 0, \\
\infty & \text{otherwise},
\end{cases}
\tag{2.3}
\]

the **inner dilatation** of \( f \) at \( x \) by
\[
K_I(x) = K_I(x, f) = \begin{cases} 
\frac{|J(x,f)|}{\|f'(x)\|^n} & \text{if } J(x,f) \neq 0, \\
1 & \text{if } f'(x) = 0, \\
\infty & \text{otherwise},
\end{cases}
\tag{2.4}
\]

Note that, see, e.g., Section 1.2.1 in [15],
\[
K_O(x, f) \leq K_I^{n-1}(x, f) \quad \text{and} \quad K_I(x, f) \leq K_O^{n-1}(x, f), \tag{2.5}
\]
in particular, \( K_O(x, f) < \infty \) a.e. if and only if \( K_I(x, f) < \infty \) a.e. The latter is equivalent to the condition that a.e. either \( \det f'(x) > 0 \) or \( f'(x) = 0 \).

Recall that a homeomorphism \( f \) between domains \( D \) and \( D' \) in \( \mathbb{R}^n, n \geq 2 \), is called of **finite distortion** if \( f \in W^{1,1}_{\text{loc}} \) and
\[
\|f'(x)\|^n \leq K(x) \cdot J_f(x) \tag{2.6}
\]
for some a.e. finite function \( K \). The term is due to Tadeusz Iwaniec. In other words, (2.6) just means that dilatations \( K_O(x, f) \) and \( K_I(x, f) \) are finite a.e.

In view of (2.5), the next statement says on a stronger modulus estimate than the obtained in [12], Theorem 4.1, in terms of the outer dilatation \( K_O(x, f) \).

**Theorem 2.1.** Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n, n \geq 3 \), and let \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) be a nondecreasing function such that, for some \( t_* \in \mathbb{R}^+ \),
\[
\int_{t_*}^{\infty} \left( \frac{t}{\varphi(t)} \right)^{\frac{1}{n-2}} dt < \infty. \tag{2.7}
\]

Then each homeomorphism \( f : D \to D' \) of finite distortion in the class \( W^{1,\varphi}_{\text{loc}} \) is a lower \( Q \)-homeomorphism at every point \( x_0 \in D' \) with \( Q(x) = [K_I(x, f)]^{\frac{1}{n-2}} \).
Proof. Let $B$ be a (Borel) set of all points $x \in D$ where $f$ has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun’s theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [7], we see that $B$ is the union of a countable collection of Borel sets $B_l$, $l = 1, 2, \ldots$, such that $f_l = f|B_l$ are bi-Lipschitz homeomorphisms, see, e.g., 3.2.2 as well as 3.1.4 and 3.1.8 in [7]. With no loss of generality, we may assume that the $B_l$ are mutually disjoint. Denote also by $B_*$ the rest of all points $x \in D$ where $f$ has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \cup B_*)$ has Lebesgue measure zero, see Theorem 1 in [11]. Hence $A_S(B_0) = 0$ for a.e. hypersurface $S$ in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in D$, see Theorem 2.11 in [9] or Theorem 9.1 in [14]. Thus, by Corollary 4 in [11] $A_S^*(f(B_0)) = 0$ as well as $A_S^*(f(B_*)) = 0$ for a.e. $S_r$ where $S_r^* = f(S_r)$.

Let $\Gamma$ be the family of all intersections of the spheres $S_r$, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup_{x \in D} |x - x_0|$, with the domain $D$. Given $g_*$ \in adm $f(\Gamma)$ such that $g_* \equiv 0$ outside of $f(D)$, set $g \equiv 0$ outside of $D$ and on $D \setminus B$ and, moreover,

$$
g(x) := \Lambda(x) \cdot g_*(f(x)) \quad \text{for } x \in B\]
$$

where

$$
\Lambda(x) = \left[ J_f(x) \cdot K_l^{\frac{1}{n-1}}(x, f) \right]^{\frac{1}{n-1}} = \left[ \det f'(x) \right]^{\frac{1}{n-1}} = \left[ \lambda_2, \ldots, \lambda_n \right]^{\frac{1}{n-1}} \geq \left[ J_{n-1}(x) \right]^{\frac{1}{n-1}} \quad \text{for a.e. } x \in B;
$$

here as usual $\lambda_n \geq \ldots \geq \lambda_1$ are principal dilatation coefficients of $f'(x)$, see, e.g., Section I.4.1 in [15], and $J_{n-1}(x)$ is the $(n - 1)$-dimensional Jacobian of $f|S_r$ at $x$ where $r = |x - x_0|$, see Section 3.2.1 in [7].

Arguing piecewise on $B_l$, $l = 1, 2, \ldots$, and taking into account Kirszbraun’s theorem, by Theorem 3.2.5 on the change of variables in [7], we have that

$$
\int_{S_r} g^{n-1} \, dA \geq \int_{S_r^*} g_*^{n-1} \, dA \geq 1
$$

for a.e. $S_r$ and, thus, $g \in ext \, adm \, \Gamma$.

The change of variables on each $B_l$, $l = 1, 2, \ldots$, see again Theorem 3.2.5 in [7], and countable additivity of integrals give also the estimate

$$
\int_D \frac{g^{n}(x)}{K_l^{\frac{1}{n-1}}(x)} \, dm(x) \leq \int_{f(D)} g_*^{n}(x) \, dm(x)
$$

and the proof is complete. \qed
Corollary 2.1. Each homeomorphism \( f \) with finite distortion in \( \mathbb{R}^n, n \geq 3 \), of the class \( W^{1,p}_{\text{loc}} \) for \( p > n - 1 \) is a lower \( Q \)-homeomorphism at every point \( x_0 \in \overline{D} \) with \( Q = K_I^{1/(n-1)} \).

Moreover, by Corollary 5 in [11] on a connection between lower and ring \( Q \)-homeomorphisms, we also obtain the following consequence of Theorem 2.1.

Theorem 2.2. Let \( f : D \to \mathbb{R}^n, n \geq 3 \), be a homeomorphism with \( K_I \in L^1_{\text{loc}} \) in \( W^{1,\varphi}_{\text{loc}} \) where \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a nondecreasing function with condition (2.7). Then \( f \) is a ring \( Q \)-homeomorphism at every point \( x_0 \in D \) with \( Q = K_I \).

In view of (2.5), the latter statement says on a stronger modulus estimate than the obtained in [11], Corollary 9, in terms of the outer dilatation \( K_O \).

Remark 2.1. By Remark 8 in [11] the conclusion of Theorem 2.2 is valid if \( K_I \) is integrable only on almost all spheres of small enough radii centered at \( x_0 \) assuming that the function \( K_I \) is extended by zero outside of \( D \).

3. On finitely bi–Lipschitz mappings

Given an open set \( \Omega \subseteq \mathbb{R}^n, n \geq 2 \), following Section 5 in [9], see also Section 10.6 in [14], we say that a mapping \( f : \Omega \to \mathbb{R}^n \) is finitely bi-Lipschitz if

\[
0 < l(x, f) \leq L(x, f) < \infty \quad \forall \ x \in \Omega
\]

where

\[
L(x, f) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}
\]

and

\[
l(x, f) = \liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|}.
\]

By the classic Rademacher–Stepanov theorem, we obtain from the right hand inequality in (3.1) that finitely bi-Lipschitz mappings are differentiable a.e. and from the left hand inequality in (3.1) that \( J_f(x) \neq 0 \) a.e. Moreover, such mappings have \((N)\)–property with respect to each Hausdorff measure, see, e.g., either Lemma 5.3 in [9] or Lemma 10.6 [14]. Thus, the proof of the following theorems is perfectly similar to one of Theorem 2.1 and hence we omit it, cf. also similar but weaker Corollary 5.15 in [9] and Corollary 10.10 in [14] formulated in terms of the outer dilatation \( K_O \).

Theorem 3.1. Every finitely bi-Lipschitz homeomorphism \( f : \Omega \to \mathbb{R}^n, n \geq 2 \), is a lower \( Q \)-homeomorphism with \( Q = K_I^{1/(n-1)} \).
References


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