Annals of the University of Bucharest (mathematical series)
4 (LXII) (2013), 213–227

A penalized viscoplastic contact problem with unilateral constraints

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Abstract - In this paper we study a mathematical model which describes the quasistatic contact between a viscoplastic body and a foundation. The contact is frictionless and is modelled with a new and nonstandard condition which involves both normal compliance, unilateral constraint and memory effects. We present a penalization method in the study of this problem. We start by introducing the penalized problem, then we prove its unique solvability as well as the convergence of its solution to the solution of the original problem, as the penalization parameter converges to zero.

Key words and phrases : viscoplastic material, frictionless contact, unilateral constraint, weak solution.

Mathematics Subject Classification (2010) **:** 74G25, 74G30, 74M15.

1. Introduction

The aim of this paper is to study a frictionless contact problem for rate-type viscoplastic materials within the framework of the Mathematical Theory of Contact Mechanics. We model the material's behavior with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor. Here $\mathcal{E}$ is a linear operator which describes the elastic properties of the material and $\mathcal{G}$ is a nonlinear constitutive function which describes its viscoplastic behavior. In (1.1) and everywhere in this paper the dot above a variable represents the derivative with respect to the time variable t. Quasistatic frictionless contact problems for materials of the form (1.1) have been considered in [2, 7, 10, 12] and the references therein. In [7, 10] the contact was modelled with both the Signorini and the normal compliance condition which describe a rigid and an elastic foundation, respectively. In [2, 12] the contact was modelled with normal compliance and unilateral constraint. This condition, introduced for the first time in [8], models an elastic-rigid behavior of the foundation.

The present paper represents a continuation of the problems studied in [3, 6]. There, a model which involves a contact condition with normal compliance, unilateral constraint and memory term was considered. This condition takes into account both the deformability, the rigidity, and the memory effects of the foundation. An existence and uniqueness result was proved and the contact process was studied on an unbounded interval of time which implies the use of the framework of Fréchet spaces of continuous functions, instead of that of the classical Banach spaces of continuous functions defined on a bounded interval of time. The aim of this work is to provide a penalization method in the study of the contact model in [6]. Penalization methods in the study of contact problems were used by many authors, mainly for numerical reasons. The main ingredient of these methods arises in the fact that they remove the constraints by considering penalized problems defined on the whole space; these approximative problems have a unique solution which converges to the solution of the original problem, as the penalization parameter converges to zero.

The rest of the paper is structured as follows. In Section 2 we present the notation we shall use as well as some preliminary material. In Section 3 we describe the model of the contact process, list the assumptions on the data and derive the variational formulation of the problem. Then we state an existence and uniqueness result, Theorem 3.1, proved in [6]. In Section 4 we present the weak solvability of the penalized problem then we state and prove our main convergence result.

2. Notations and Preliminaries

Everywhere in this paper we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, +\infty)$. For a given $r \in \mathbb{R}$ we denote by r^+ its positive part, i.e. $r = max\,\{r, 0\}$. Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that meas $(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i / \partial x_j$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\begin{aligned} &\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \\ &\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d. \end{aligned}$$

In addition, we use standard notation for the Lebesgue and Sobolev spaces associated to Ω and Γ and, moreover, we consider the spaces

$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1 \,\}$, $Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d\times d} : \tau_{ij} = \tau_{ji} \,\}$.

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} \in H^1(\Omega)^d.$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $\operatorname{meas}(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\, \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.1}$$

Also, for a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential traces, i.e. $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Moreover, we recall that the divergence operator is defined by the equality $\operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})$ and, finally, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \boldsymbol{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \quad \forall\, \boldsymbol{v} \in V. \tag{2.2}$$

Finally, we consider the space of fourth order tensor fields

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) \ :\ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i, j, k, l \le d\ \}.$$

This is a real Banach space with the norm $\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1 \le i,j,k,l \le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$. Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \qquad \forall\, \mathcal{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{2.3}$$

For each Banach space X we use the notation $C(\mathbb{R}_+; X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values in X. For a subset $K \subset X$ we still use the symbol $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K. It is well known that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, i.e. as a complete metric space in which the corresponding topology is induced by a countable family of seminorms.

Details can be found in [5] and [9], for instance. Here we restrict ourseleves to recall that the convergence of a sequence $(x_k)_k$ to the element x, in the space $C(\mathbb{R}_+; X)$, can be described as follows:

$$\begin{cases} x_k \to x \quad \text{in } C(\mathbb{R}_+; X) \ \text{ as } \ k \to \infty \ \text{ if and only if} \\ \max\limits_{r\in[0,n]} \|x_k(r) - x(r)\|_X \to 0 \ \text{ as } k \to \infty, \quad \text{for all } \ n \in \mathbb{N}^*. \end{cases} \tag{2.4}$$

3. Problem statement

The physical setting is as follows. A viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\mathrm{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that the contact process is quasistatic and we study it in the interval of time $\mathbb{R}_+ = [0, \infty)$. Then, the classical formulation of the contact problem we consider in this paper is the following.

Problem $\mathcal{P}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{aligned} & u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) + \xi(t) \le 0, \\ & (u_\nu(t) - g)\big(\sigma_\nu(t) + p(u_\nu(t)) + \xi(t)\big) = 0, \\ & 0 \le \xi(t) \le \int_0^t b(t-s)\, u_\nu^+(s)\, ds, \\ & \xi(t) = 0 \ \text{ if } \ u_\nu(t) < 0, \\ & \xi(t) = \int_0^t b(t-s)\, u_\nu^+(s)\, ds \ \text{ if } \ u_\nu(t) > 0 \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{3.6}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \qquad \text{in} \quad \Omega. \tag{3.7}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equation (3.1) represents the viscoplastic constitutive law of the material already introduced in Section 1. Equation (3.2) is the equilibrium equation in which Div denotes the divergence operator for tensor valued functions. Conditions (3.3) and (3.4) are the displacement and traction boundary conditions, respectively. Condition (3.5) shows that the tangential stress on the contact surface, denoted $\boldsymbol{\sigma}_\tau$, vanishes. We use it here since we assume that the contact process is frictionless. Condition (3.6) represents the contact condition with normal compliance, unilateral constraint and memory term, in which σ_ν denotes the normal stress, u_ν is the normal displacement, $g \geq 0$ and p, b are given functions. This condition was first introduced in [6] and, in the case when b vanishes, was used in [8, 11], for instance. Finally, (3.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

Next, we list the assumptions on the data, present the variational formulation of the problem $\mathcal{P}$ and then we state and prove its unique weak solvability. To this end, we assume that the elasticity tensor $\mathcal{E}$, the nonlinear constitutive function $\mathcal{G}$ and the normal compliance function p satisfy the following conditions.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega),\ 1 \leq i, j, k, l \leq d. \\
\text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that} \\
\qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{E}} \|\boldsymbol{\tau}\|^2 \ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega,
\end{array}\right. \tag{3.8}$$

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
\qquad \|\mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\
\qquad\quad \leq L_{\mathcal{G}} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \right) \\
\qquad\quad\ \ \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } Q
\end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{l}
\text{(a) } p : \mathbb{R} \to \mathbb{R}_+. \\
\text{(b) There exists } L_p > 0 \text{ such that} \\
\qquad |p(r_1) - p(r_2)| \leq L_p |r_1 - r_2| \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\
\text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \geq 0 \quad \forall\, r_1,\, r_2 \in \mathbb{R}. \\
\text{(d) } p(r) = 0 \ \text{ for all } r < 0.
\end{array}\right. \tag{3.10}$$

Moreover, the densities of body forces, surface tractions and the memory function are such that

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{3.11}$$

$$b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)), \qquad b(t, \boldsymbol{x}) \geq 0 \tag{3.12}$$

Finally, the initial data verifies

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q. \tag{3.13}$$

We introduce the set of admissible displacements U given by

$$U = \{\, \boldsymbol{v} \in V : v_\nu \leq g \text{ on } \Gamma_3 \,\}. \tag{3.14}$$

Next, using the Riesz representation theorem we define the operators $P : V \to V$, $\mathcal{B} : C(\mathbb{R}_+, V) \to C(\mathbb{R}_+, L^2(\Gamma_3))$ and the function $f : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V, \tag{3.15}$$

$$(\mathcal{B}\boldsymbol{u}(t), \xi)_{L^2(\Gamma_3)} = \Big(\int_0^t b(t-s)\, u_\nu^+(s)\, ds, \xi \Big)_{L^2(\Gamma_3)} \tag{3.16}$$
$$\forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \xi \in L^2(\Gamma_3),\ t \in \mathbb{R}_+,$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\, da \tag{3.17}$$
$$\forall\, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+,$$

In order to derive the variational formulation of the Problem $\mathcal{P}$ we introduce the operator $\mathcal{S}$ by the following lemma.

Lemma 3.1. *Assume that* (3.9) *and* (3.13) *hold. Then, for each function* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *there exists a unique function* $\mathcal{S}\boldsymbol{u} \in C(\mathbb{R}_+; Q)$ *such that*

$$\mathcal{S}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}\boldsymbol{u}(s) + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds + \boldsymbol{\sigma}_0 - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0) \quad \forall\, t \in \mathbb{R}_+. \tag{3.18}$$

Moreover, the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ *satisfies the following condition: for every* $n \in \mathbb{N}$ *there exists* $k_n > 0$ *such that,* $\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V)$, $\forall\, t \in [0, n]$,

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_Q \leq k_n \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds. \tag{3.19}$$

The variational formulation of Problem $\mathcal{P}$ is the following.

Problem $\mathcal{P}^V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to U$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}\boldsymbol{u}(t) \quad \forall\, t \in \mathbb{R}_+, \tag{3.20}$$

$$\begin{gathered}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ +(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - u_\nu^+(t))_{L^2(\Gamma_3)} + (P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall\, \boldsymbol{v} \in U,\ \forall\, t \in \mathbb{R}_+.\end{gathered} \tag{3.21}$$

The proof of Lemma 3.1 as well as the variational formulation $\mathcal{P}^V$ were obtained in [6]. Note that (3.20) is a consequence of (3.1), (3.7) and (3.18), while (3.21) can be easily obtained by using integrations by parts, (3.2)–(3.5) and notation (3.14)–(3.18). The unique weak solvability of Problem $\mathcal{P}$ follows from the following result.

Theorem 3.1. *Assume that (3.8)–(3.13) hold. Then Problem $\mathcal{P}^V$ has a unique solution, which satisfies $\boldsymbol{u} \in C(\mathbb{R}_+; U)$ and $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.*

The proof of Theorem 3.1 was given in [6], based on an abstract result provided by [12, 13].

4. A penalization result

In this section we introduce a penalized contact problem $\mathcal{P}_\mu$ and we prove that its unique weak solution converges to the weak solution of problem $\mathcal{P}$.

Let q be a function which satisfies

$$\left\{\begin{array}{l}\text{(a) } q : [g, +\infty[\to \mathbb{R}_+. \\ \text{(b) There exists } L_q > 0 \text{ such that} \\ \qquad |q(r_1) - q(r_2)| \leq L_q |r_1 - r_2| \quad \forall\, r_1,\, r_2 \geq g. \\ \text{(c) } (q(r_1) - q(r_2))(r_1 - r_2) > 0 \quad \forall\, r_1,\, r_2 \geq g,\, r_1 \neq r_2. \\ \text{(d) } q(g) = 0.\end{array}\right. \tag{4.1}$$

Let $\mu > 0$ and consider the function p_μ defined by

$$p_\mu(r) = \left\{\begin{array}{ccc} p(r) & \text{if} & r \leq g, \\ \frac{1}{\mu}\, q(r) + p(g) & \text{if} & r > g. \end{array}\right. \tag{4.2}$$

We deduce from (4.1) and (4.2) that the function p_μ satisfies condition (3.10), i.e.

$$\left\{\begin{array}{l}\text{(a) } p_\mu : \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) There exists } L_{p_\mu} > 0 \text{ such that} \\ \qquad |p_\mu(r_1) - p_\mu(r_2)| \leq L_{p_\mu} |r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p_\mu(r_1) - p_\mu(r_2))(r_1 - r_2) \geq 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p_\mu(r) = 0 \ \text{ for all } r < 0.\end{array}\right. \tag{4.3}$$

This allows us to consider the operator $P_\mu : V \to V$ defined by

$$(P_\mu \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\mu(u_\nu) v_\nu \, da \quad \forall\, \boldsymbol{u},\, \boldsymbol{v} \in V \tag{4.4}$$

and we note that P_μ is a monotone Lipschitz continuous operator.

With these notation, we consider the following contact problem.

Problem $\mathcal{P}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}_\mu : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\dot{\boldsymbol{\sigma}}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\mu(t)) + \mathcal{G}(\boldsymbol{\sigma}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t))) \quad \text{in} \quad \Omega, \tag{4.5}$$

$$\operatorname{Div} \boldsymbol{\sigma}_\mu(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{4.6}$$

$$\boldsymbol{u}_\mu(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{4.7}$$

$$\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4.8}$$

$$\boldsymbol{\sigma}_{\mu\tau}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{4.9}$$

for all $t \in \mathbb{R}_+$, *there exists* $\xi : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *which satisfies*

$$\left.\begin{array}{l} \sigma_{\mu\nu}(t) + p_\mu(u_{\mu\nu}(t)) + \xi(t) = 0, \\ 0 \le \xi(t) \le \displaystyle\int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds, \\ \xi(t) = 0 \ \text{ if } \ u_{\mu\nu}(t) < 0, \\ \xi(t) = \displaystyle\int_0^t b(t-s)\, u_{\mu\nu}^+(s)\, ds \ \text{ if } \ u_{\mu\nu}(t) > 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{4.10}$$

for all $t \in \mathbb{R}_+$ *and, moreover,*

$$\boldsymbol{u}_\mu(0) = \boldsymbol{u}_0,\, \boldsymbol{\sigma}_\mu(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{4.11}$$

Note that here and below $u_{\mu\nu}$ represents the normal component of the displacement field $\boldsymbol{u}_\mu$ and $\sigma_{\mu\nu}$, $\boldsymbol{\sigma}_{\mu\tau}$ represent the normal and tangential components of the stress tensor $\boldsymbol{\sigma}_\mu$, respectively. The equations and boundary conditions in problem (4.5)–(4.11) have a similar interpretation as those in problem (3.1)–(3.7). The difference arises in the fact that here we replace the contact condition with normal compliance, memory term and unilateral constraint (3.6) with the contact condition with normal compliance and memory term (4.10). In this condition μ represents a penalization parameter which may be interpreted as a deformability coefficient of the foundation, and then $\frac{1}{\mu}$ is the surface stiffness coefficient.

Using notation (3.17), (3.16) and (4.4) by similar arguments as in the case of Problem $\mathcal{P}$ we obtain the following variational formulation of Problem $\mathcal{P}_\mu$.

Problem $\mathcal{P}^V_\mu$. *Find a displacement field $\boldsymbol{u}_\mu : \mathbb{R}_+ \to V$ and a stress field $\boldsymbol{\sigma}_\mu : \mathbb{R}_+ \to Q$ such that, for all $t \in \mathbb{R}_+$,*

$$\boldsymbol{\sigma}_\mu(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) + \mathcal{S}\boldsymbol{u}_\mu(t) \quad \forall\, t \in \mathbb{R}_+, \tag{4.12}$$

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)))_Q \\ +(\mathcal{B}\boldsymbol{u}_\mu(t), v_\nu^+ - u_{\mu\nu}^+(t))_{L^2(\Gamma_3)} + (P_\mu \boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \quad \forall\, \boldsymbol{v} \in V,\ \forall\, t \in \mathbb{R}_+.\end{aligned} \tag{4.13}$$

We have the following existence, uniqueness and convergence result.

Theorem 4.1. *Assume that* (3.8) – (3.13) *and* (4.1) *hold. Then*

a) For each $\mu > 0$ there exists a unique solution $\boldsymbol{u}_\mu \in V$ to Problem $\mathcal{P}^V_\mu$.

b) The solution $\boldsymbol{u}_\mu$ of Problem $\mathcal{P}^V_\mu$ converges strongly to the solution $\boldsymbol{u}$ of Problem $\mathcal{P}^V$, that is

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V + \|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \tag{4.14}$$

as $\mu \to 0$, for all $t \in \mathbb{R}_+$.

Note that the convergence (4.14) above is understood in the following sense: for all $t \in \mathbb{R}_+$ and for every sequence $\{\mu_n\} \subset \mathbb{R}_+$ converging to 0 as $n \to \infty$ we have $\boldsymbol{u}_{\mu_n}(t) \to \boldsymbol{u}(t)$ in V and $\boldsymbol{\sigma}_{\mu_n}(t) \to \boldsymbol{\sigma}(t)$ in Q as $n \to \infty$.

The proof of Theorem 4.1 is carried out in several steps that we present in what follows. To this end we assume below that (3.8)–(3.13) and (4.1) hold. Let $\mu > 0$. We consider the auxiliary problem of finding a displacement field $\widetilde{\boldsymbol{u}}_\mu : \mathbb{R}_+ \to V$ such that, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ +(\mathcal{B}\boldsymbol{u}(t), v_\nu^+ - \widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)} + (P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t))_V \\ \geq (\boldsymbol{f}(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t))_V \ \ \forall\, \boldsymbol{v} \in V.\end{aligned} \tag{4.15}$$

This problem is an intermediate problem between (4.13) and (3.21), since here $\mathcal{S}\boldsymbol{u}(t)$, $\mathcal{B}\boldsymbol{u}(t)$ are knowns, taken from the problem $\mathcal{P}^V$.

We have the following existence and uniqueness result.

Lemma 4.1. *There exists a unique function $\widetilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ which satisfies* (4.15)*, for all $t \in \mathbb{R}_+$.*

Proof. We define the operator $A_\mu : V \to V$ and the function $\widetilde{\boldsymbol{f}} : \mathbb{R}_+ \to V$ by equalities

$$(A_\mu \boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P_\mu \boldsymbol{u}, \boldsymbol{v})_V \tag{4.16}$$

$$(\widetilde{\boldsymbol{f}}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V - (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q - (\mathcal{B}\boldsymbol{u}(t), v_\nu^+)_{L^2(\Gamma_3)}, \tag{4.17}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V,\ t \in \mathbb{R}_+$. We note that (3.12), (3.11), (3.17) and (3.18) yield $\widetilde{\boldsymbol{f}} \in C(\mathbb{R}_+; V)$.

Let $t \in \mathbb{R}_+$. Based on (4.16)–(4.17), it is easy to see that (4.15) is equivalent with the nonlinear variational inequality of the first kind

$$(A_\mu \widetilde{\boldsymbol{u}}_\mu(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t))_V \geq (\widetilde{\boldsymbol{f}}(t), \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\mu(t))_V \quad \forall\, \boldsymbol{v} \in V. \tag{4.18}$$

Next, by (3.8) and the properties of operator P_μ it follows that A_μ is a strongly monotone and Lipschitz continuous operator. Therefore, using standard arguments on variational inequalities we deduce that there exists a unique solution $\widetilde{\boldsymbol{u}}_\mu \in C(\mathbb{R}_+; V)$ for (4.18), which concludes the proof. □

We proceed with the following weak convergence result.

Lemma 4.2. *As* $\mu \to 0$,

$$\widetilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \boldsymbol{u}(t) \quad in \ \ V,$$

for all $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$. We take $\boldsymbol{v} = \boldsymbol{0}$ in (4.15) to obtain

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \leq (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V - (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q\\ &-(\mathcal{B}\boldsymbol{u}(t), \widetilde{u}^+_{\mu\nu}(t))_{L^2(\Gamma_3)} - (P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V\end{aligned} \tag{4.19}$$

On the other hand, the properties (4.3) yield $(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V \geq 0$, and from (4.19) we deduce that

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \leq (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V\\ &\quad -(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{B}\boldsymbol{u}(t), \widetilde{u}^+_{\mu\nu}(t))_{L^2(\Gamma_3)}.\end{aligned} \tag{4.20}$$

From (3.8) we obtain that

$$\|\widetilde{\boldsymbol{u}}_\mu(t)\|_V \leq c\,(\|\boldsymbol{f}(t)\|_V + \|\mathcal{S}\boldsymbol{u}(t)\|_V + \|\mathcal{B}\boldsymbol{u}(t)\|_{L^2(\Gamma_3)}). \tag{4.21}$$

Note that here and below c is a constant which does not depend on μ and t and whose value can change from line to line. This inequality shows that the sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu \subset V$ is bounded. Hence, there exists a subsequence of the sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$, still denoted $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$, and an element $\widetilde{\boldsymbol{u}}(t) \in V$ such that

$$\widetilde{\boldsymbol{u}}_\mu(t) \rightharpoonup \widetilde{\boldsymbol{u}}(t) \ \text{ in } V \quad \text{as } \mu \to 0. \tag{4.22}$$

Next we study the properties of the element $\widetilde{\boldsymbol{u}}(t)$. It follows from (4.19) that

$$\begin{aligned}(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V &\leq (\boldsymbol{f}(t), \widetilde{\boldsymbol{u}}_\mu(t))_V - (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q\\ &\quad -(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{B}\boldsymbol{u}(t), \widetilde{u}^+_{\mu\nu}(t))_{L^2(\Gamma_3)}\end{aligned}$$

and, since $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is a bounded sequence in V, we deduce that

$$(P_\mu \widetilde{\boldsymbol{u}}_\mu(t), \widetilde{\boldsymbol{u}}_\mu(t))_V \le c.$$

This implies that $\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t))\widetilde{u}_{\mu\nu}(t)\,da \le c$ and, since $\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t))g\,da \ge 0$, it follows that

$$\int_{\Gamma_3} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le c. \tag{4.23}$$

We consider now the measurable subsets of Γ_3 defined by

$$\Gamma_{31} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \widetilde{u}_{\mu\nu}(t)(\boldsymbol{x}) \le g \,\}, \quad \Gamma_{32} = \{\, \boldsymbol{x} \in \Gamma_3 \,:\, \widetilde{u}_{\mu\nu}(t)(\boldsymbol{x}) > g \,\}. \tag{4.24}$$

Clearly, both Γ_{31} and Γ_{32} depend on t and μ but, for simplicity, we do not indicate explicitly this dependence. We use (4.23) to write

$$\int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da + \int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le c$$

and, since $\int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t))\widetilde{u}_{\mu\nu}(t)\,da \ge 0$, we obtain

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le \int_{\Gamma_{31}} p_\mu(\widetilde{u}_{\mu\nu}(t))g\,da + c.$$

Thus, taking into account that $p_\mu(r) = p(r)$ for $r \le g$, by the monotonicity of the function p we can write

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le \int_{\Gamma_{31}} p(\widetilde{u}_{\mu\nu}(t))g\,da + c \le \int_{\Gamma_3} p(g)g\,da + c.$$

Therefore, we deduce that

$$\int_{\Gamma_{32}} p_\mu(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le c. \tag{4.25}$$

We use now the definitions (4.2) and (4.24) to see that, a.e on Γ_{32}, we have

$$p_\mu(\widetilde{u}_{\mu\nu}(t)) = \frac{1}{\mu}\, q(\widetilde{u}_{\mu\nu}(t)) + p(g), \quad p(g)(\widetilde{u}_{\mu\nu}(t) - g) > 0.$$

Consequently, the inequality (4.25) yields

$$\int_{\Gamma_{32}} q(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t) - g)\,da \le c\mu. \tag{4.26}$$

Next, we consider the function defined by

$$\widetilde{p}:\mathbb{R}\to\mathbb{R}_+ \quad \widetilde{p}(r)=\begin{cases} 0 & \text{if} \quad r\le g,\\ q(r) & \text{if} \quad r>g\end{cases}$$

and we note that by (4.1) it follows that $\widetilde{p}$ is a continuous increasing function and, moreover,

$$\widetilde{p}(r)=0 \qquad \text{iff} \qquad r\le g. \tag{4.27}$$

We use (4.26), equality $q(\widetilde{u}_{\mu\nu}(t))=\widetilde{p}(\widetilde{u}_{\mu\nu}(t))$ a.e on Γ_{32} and (4.24) to deduce that

$$\int_{\Gamma_3}\widetilde{p}(\widetilde{u}_{\mu\nu}(t))(\widetilde{u}_{\mu\nu}(t)-g)^+\le c\mu,$$

where $(\widetilde{u}_{\mu\nu}(t)-g)^+$ denotes the positive part of $\widetilde{u}_{\mu\nu}(t)-g$. Therefore, passing to the limit as $\mu\to 0$, by using (4.22) as well as compactness of the trace operator we find that

$$\int_{\Gamma_3}\widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+\,da\le 0.$$

Since the integrand $\widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+$ is positive a.e on Γ_3, the last inequality yields $\widetilde{p}(\widetilde{u}_\nu(t))(\widetilde{u}_\nu(t)-g)^+=0$ a.e on Γ_3 and, using (4.27) and definition (3.14) we conclude that

$$\widetilde{\boldsymbol{u}}(t)\in U. \tag{4.28}$$

Since $\boldsymbol{v}\in U$ we have $p_\mu(v_\nu)=p(v_\nu)$ a.e. on Γ_3. Taking into account this equality and the monotonicity of the function p_μ we have

$$p(v_\nu)(v_\nu-\widetilde{u}_{\mu\nu}(t))\ge p_\mu(\widetilde{u}_{\mu\nu}(t))(v_\nu-\widetilde{u}_{\mu\nu}(t)) \quad \text{a.e. on } \Gamma_3$$

and, therefore, by using (4.4) we obtain

$$(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\ge(P_\mu\widetilde{\boldsymbol{u}}_\mu(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V. \tag{4.29}$$

Then, using (4.29) and (4.15) we find that

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q\\ +(\mathcal{B}\boldsymbol{u}(t),v_\nu^+-\widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)}+(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\ge(\boldsymbol{f}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}_\mu(t))_V\end{aligned} \tag{4.30}$$

for all $\boldsymbol{v}\in U$. We pass to the lower limit in (4.30) and use (4.22) to obtain

$$\begin{aligned}(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t)))_Q+(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t)))_Q\\ +(\mathcal{B}\boldsymbol{u}(t),v_\nu^+-\widetilde{u}_\nu^+(t))_{L^2(\Gamma_3)}+(P\boldsymbol{v},\boldsymbol{v}-\widetilde{\boldsymbol{u}}(t))_V\ge(\boldsymbol{f}(t),\boldsymbol{v}-\widetilde{\boldsymbol{u}}(t))_V\end{aligned} \tag{4.31}$$

for all $\boldsymbol{v}\in U$. Next, we take $\boldsymbol{v}=\widetilde{\boldsymbol{u}}(t)$ in (3.21) and $\boldsymbol{v}=\boldsymbol{u}(t)$ in (4.31). Then, adding the resulting inequalities we find that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t))-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q\le 0.$$

Using (3.8), the above inequality implies that $\widetilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t)$. It follows from here that the whole sequence $\{\widetilde{\boldsymbol{u}}_\mu(t)\}_\mu$ is weakly convergent to the element $\boldsymbol{u}(t) \in V$, which concludes the proof. □

We proceed with the following strong convergence result.

Lemma 4.3. *As* $\mu \to 0$,

$$\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0,$$

for all $t \in \mathbb{R}_+$.

Proof. Let $t \in \mathbb{R}_+$ and $\mu > 0$. Using (3.8) we write

$$\begin{aligned} m_{\mathcal{E}}\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V^2 &\le (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \\ &= (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q - (\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q. \end{aligned}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (4.30) to obtain

$$\begin{aligned} &-(\mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \le (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &+(\mathcal{B}\boldsymbol{u}(t), u_\nu^+(t) - \widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)} + (P\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V - (\boldsymbol{f}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \end{aligned}$$

and, therefore, combining the above inequalities we find that

$$\begin{aligned} m_{\mathcal{E}}\|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V^2 &\le (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &+(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q + (P\boldsymbol{u}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V \\ &+(\mathcal{B}\boldsymbol{u}(t), u_\nu^+(t) - \widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)} - (\boldsymbol{f}(t), \boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}_\mu(t))_V. \end{aligned}$$

We pass to the upper limit in this inequality and use Lemma 4.2 to conclude the proof. □

We are now in position to provide the proof of Theorem 4.1.

Proof. Let $t \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ be such that $t \in [0, n]$. Let also $\mu > 0$. Next, we take $\boldsymbol{v} = \boldsymbol{u}_\mu(t)$ in (4.15) and $\widetilde{\boldsymbol{u}}_\mu(t)$ in (4.13). Then adding the resulting inequalities and using the monotonicity of the operator P_μ we deduce that

$$\begin{aligned} &(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \mathcal{E}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &\quad\le (\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) - \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_\mu(t)))_Q \\ &\quad+(\mathcal{B}\boldsymbol{u}(t) - \mathcal{B}\boldsymbol{u}_\mu(t), u_{\mu\nu}^+(t) - \widetilde{u}_{\mu\nu}^+(t))_{L^2(\Gamma_3)} \end{aligned}$$

and, therefore,

$$\|\boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{c}{m_{\mathcal{E}}}\left(\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{u}_\mu(t)\|_Q + \|\mathcal{B}\boldsymbol{u}(t) - \mathcal{B}\boldsymbol{u}_\mu(t)\|_{L^2(\Gamma_3)}\right). \tag{4.32}$$

We use (4.32) to find that

$$\|\boldsymbol{u}_\mu(t) - \widetilde{\boldsymbol{u}}_\mu(t)\|_V \le \frac{r_n}{m_{\mathcal{E}}} \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{u}_\mu(s)\|_V \, ds$$

where $r_n = k_n + c_0^2 \max\limits_{r\in[0,n]} \|b(r)\|_{L^2(\Gamma_3)}$. It follows from here that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{E}}} \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V ds$$

and, using a Gronwall argument, we obtain

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{E}}} \int_0^t e^{\frac{r_n}{m_{\mathcal{E}}}(t-s)} \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds.$$

Note that $e^{\frac{r_n}{m_{\mathcal{E}}}(t-s)} \le e^{\frac{r_n}{m_{\mathcal{E}}}t} \le e^{\frac{nr_n}{m_{\mathcal{E}}}}$ for all $s \in [0,t]$ and we deduce that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \le \|\widetilde{\boldsymbol{u}}_\mu(t) - \boldsymbol{u}(t)\|_V + \frac{r_n}{m_{\mathcal{E}}} e^{\frac{nr_n}{m_{\mathcal{E}}}} \int_0^t \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds. \quad (4.33)$$

On the other hand, by estimate (4.21), Lemma 4.3 and Lebesgue's convergence Theorem it follows that

$$\int_0^t \|\widetilde{\boldsymbol{u}}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds \to 0 \qquad \text{as} \quad \mu \to 0. \qquad (4.34)$$

We use now (4.33), (4.34) and Lemma 4.3 to see that

$$\|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t)\|_V \to 0 \qquad \text{as} \quad \mu \to 0. \qquad (4.35)$$

Next, by (3.20), (4.12), (3.8), (3.19) and (3.12) it follows that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \le c\, \|\boldsymbol{u}_\mu(t) - \boldsymbol{u}(t))\|_V + k_n \int_0^t \|\boldsymbol{u}_\mu(s) - \boldsymbol{u}(s)\|_V \, ds.$$

We use again the convergence (4.35) and Lebesque's Theorem to find that

$$\|\boldsymbol{\sigma}_\mu(t) - \boldsymbol{\sigma}(t)\|_Q \to 0 \qquad \text{as} \quad \mu \to 0. \qquad (4.36)$$

Theorem 4.1 is now a consequence of the convergences (4.35) and (4.36). □

Acknowledgments

The work of the second author was supported within the Sectoral Operational Programme for Human Resources Development 2007-2013, cofinanced by the European Social Fund, under the project POSDRU/107/1.5/S/76841 entitled *Modern Doctoral Studies: Internationalization and Interdisciplinarity*, at Babeş-Bolyai University, Cluj-Napoca. The work of the third author was supported by a Marie Curie International Research Staff Exchange Scheme Fellowship within the seventh European Community Framework Programme under Grant Agreement no. 2011-295118.

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