Multilattices in Informatics: introductory examples  
(with some remarks on modelling)  
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The present paper is motivated by the contemporary interest for the works of M. Benado concerning multilattices – a partially ordered structure generalizing lattices (see further) – shown in several articles published during 50 years after his initial contributions from 1953 – 1955. The motivation has also a personal connotation: both authors of this paper have been Benado’s students in the interval 1956-1958 – getting from him an attachment for lattices and an interiorization of mathematics (by now still alive).

The attempt made here would not be to provide a report more or less complete. Instead we try to point out to some ideas of Benado which we consider to be relevant now in Theoretical Informatics loosely described as being the mathematical basis of the field.

1 Mihail Benado was a Romanian algebraist who studied lattice theory, group theory and their interconnections. Benado was born on July 5, 1920 in Bucharest. He was a scholar at the elementary school of the Jewish Community, then at high school „Matei Basarab”, which he graduated in 1940. Because of the anti-Semitic law „Numerus Clausus” in vigour in Romania of those days, Benado could not become a student of a State University, but he attended the courses of a private university. In 1944 he became a student of the Faculty of Sciences (including divisions of mathematics, physics and chemistry) of the University of Bucharest, which he graduated in 1948, the very year in which the Faculty of Mathematics (and Physics) was created as a separate faculty. From 1948 to 1962 Benado held teaching positions at this faculty, and from 1950 to 1962 he was also a researcher at the Institute of Mathematics of the Romanian Academy. In January 1962 he resigned both positions.

Unfortunately, in the sixties Benado gradually isolated himself from the mathematical community of Romania and, excepting a rather short period in which he was with the (3-year) Pedagogical Institute of Bucharest, there are no informations about Benado's life after 1964.

Mihail Benado was a pupil of Dan Barbilian, who directed Benado's interest towards group theory and lattice theory, the latter being at that time a young branch of mathematics. The fact that Barbilian was both an algebraist and a geometer had a strong influence in the scientific career of Mihai Benado.

Two major research themes of Benado were the theory of regular products of operator groups initiated by Oleg Golovin and the Schreier-Zassenhaus and Jordan-Hölder refinement theorems in group theory and their generalizations to modular lattices. Benado introduced new concepts related to regular products, generalized several results of Golovin and generalized this theory to the lattice-theoretical level. The main concern of Benado's ample contribution to the field of refinement theorems was the passage from modular lattices to arbitrary lattices and even to partially ordered sets and to his multilattices.

Benado also introduced the monotone connections of types I, II and III, which are similar to but different from the conventional Galois connections, and characterized semilattices and semimultilattices in terms of his monotone connections. Benado also generalized several results of Glivenko and Barbilian on metric lattices. The work of Mihail Benado comprises two other big constructions. One of them is the theory of diemetric spaces, an attempt to develop an axiomatic theory of the distance between two bodies (Körper). The other construction is a general theory of partially ordered sets, even more general than the theory of multilattices. Unfortunately, these two projects have remained unfinished (the biographical data largely base don Andonie [1967][1]).
A part of our work was devoted to documentation. The first author has established – we think for the first time - a bibliography of the publications of Benado. As far as we know it is complete. A second bibliography concerns articles and books in which the ideas of Benado results in a remarkable follow up richer than initially foreseen by us. It should be observed that the works of Benado remain significant not only for the theory of multilattices [1953] but also for other fields of mathematics, e. g., for the theory of Galois connections and residuation, monotone connections or diametric spaces.

The idea to be found to Kant that a field becomes a science to the extent that it becomes mathematics should receive a continuation: and it is more applicable to the extent it is more abstract (an idea whose origins are at Gr. C. Moisil). Informatics is forcefully illustrating this fascinating process by the growing role mainly of Algebra in the fundamentals of computing. Simplifying it looks that computing becomes Computer Science mainly by its algebraization. This Note tries to illustrate this idea by means of two examples involving partially ordered structures for monoids.

Is this draft a paper on Mathematics or on Informatics? Is the building in front of us Architecture or Civil Engineering? The vital part of our computer on the desk consists in circuitry or in the invisible programs? Until we answer this question, for sure the most interesting part of our life will be exhausted.

In an earlier stage of Computer Science many computer scientists found a major interest in the lattice-theoretic approach, in Scott’s theory of continuous lattices and in the ADJ works dealing with \( \omega \)-complete sets and \( \omega \)-continuous algebras. However, many partially ordered systems of interest in Computer Science are not in fact lattices (the classical reference for lattices is Birkhoff [1967] [6]), thus requiring a more general theory of the involved ordered structures as stated by G. Markowsky in his conference “Posets, Lattices and Computer Science”, University of Maine (“many structures of interest in computer science were not naturally lattices”).

This Note aims to suggest that a convenient candidate is the theory of multilattices initiated and developed by the Romanian algebrist Mihail Benado [1953] [2], [1954] [3], [4], [1955] [5] and later on considerably enriched mostly by the Czechoslovak and Spanish mathematicians. Because of its size the litterature on multilattices remains regrettfully outside our present limits; however we are preparing a paper on some works of Benado and on their relevance in Mathematics and Computer Science.
Multilattices of M. Benado. For a poset \( P \) and for \( a, b \in P \), let \( U(a, b) \) and \( L(a, b) \) denote the sets of all upper and lower bounds of the set \( \{a, b\} \), respectively, and \( a \lor b \) be the set of all minimal elements from \( U(a, b) \) and \( a \land b \) be the set of maximal elements from \( L(a, b) \). \( P \) is a multilattice if it satisfies the condition that for \( u \in U(a, b) \), there exists \( u_1 \in a \lor b \) with \( u_1 \in L(u) = L(u, u) \) and its dual.

When comparing with lattices, roughly speaking, we see that lub (the least upper bound) which is a unique element is replaced by the set possibly void of all minimal (instead of least) upper bounds and dually.

More formally, a multilattice is defined as follows.

Let \( P = (P, \leq) \) be a partially ordered set (poset). For \( a \in P \) denote by \( [a] = \{ x \in P \mid x \leq a \} \) and \( (a] = \{ x \in P \mid x \geq a \} \), \( [a, b] \) being the usual interval, \( [a, b] = [a] \cap (b] \) while \( L(a, b) = (a] \cap (b] \) and \( U(a, b) = [a] \cap (b) \).

\( P \) is called right-directed, respectively left-directed, if for any \( a, b \in P \), the set \( U(a, b) \neq \emptyset \), respectively \( L(a, b) \neq \emptyset \). \( P \) is called directed if it is both right and left-directed. Benado [1953] [2] (in Romanian) has introduced the concept of a multilattice later on extensively studied in further works, op. cit. [1954], [1955].

**Definition.** A poset \( P \) is a multilattice if for any \( a, b, c \in P \) such that \( c \in U(a, b) \) the set \( U(a, b) \cap (c] \) has a minimal element and dually for \( c \in L(a, b) \).

A multilattice is distributive if for each \( a, b, c \in P \) the conditions \( (a \lor b) \cap (a \lor c) \neq \emptyset \), \( (a \land b) \cap (a \land c) \neq \emptyset \) imply \( b = c \).

**A syntax-oriented example.** Let \( \Sigma^* \) be the set of all strings/words over the finite alphabet \( \Sigma \), and let \( u \leq_{pf} v \) iff \( v=uv_1 \), \( u \) being called a prefix of \( v \), for \( u, v \in \Sigma^* \). Similarly, one defines the dual \( u \leq_{sf} v \) iff \( u \) is a suffix of \( v \) and \( u \leq_{fct} v \) iff \( u \) is a factor of \( v \), i.e., \( v=v_1 u v_2 \) (e.g., roma \( \leq_{pf} \) roman \( \leq_{pf} \) romania, man \( \leq_{fct} \) romania a. o.). In general, two different letters or words \( u \) and \( v \), e.g. \( u=ab \) and \( v=ba \), would not have a common pf / sf – upper bound and thus pf / sf would not result in a lattice. However they have a fct-upper bound (e.g., \( aba \) and \( bab \) since \( ab, ba \leq_{fct} aba, bab \)) but the least upper bound do not necessarily exist. Obviously, the relation \( \leq_h \), is a partial order, \( h=pf, sf, fct \).

**Remark.** If \( u \leq_h v \), then \( |u| \leq |v| \), where \( |w| \) is the length of \( w \), hence \( (\Sigma^*, \leq_h) \) is locally finite, in the sense that any interval is finite, \( h=pf, sf, fct \). Moreover, \( L=(\Sigma^*, \cdot, e, \leq_h) \) is a noncommutative monoid wrt the concatenation \( \cdot \) of words, the word of zero length \( e \) being an identity and the least/first element. Moreover, we have even a multilattice. Indeed, every locally finite poset is also chain finite. Therefore \( L \) satisfies both the
ascending and descending bounded chain conditions and thus for a common upper bound of two elements we may insert only a finite number of elements, finally terminating at a minimal element and dually.

The one-sided compatibility of the concatenation with the partial orders \( \leq_h, h=pf, sf, \) should be readily checked. For the fct partial order we have two-sided compatibility and therefore a multilattice-ordered multiplicative monoid, in general noncommutative. More general, it is sufficient for this construction to have a multiplicative structure compatible with the considered partial order.

It is possible to build up more particular still interesting examples.

**Programs semantics.** The idea of partially additive program semantics is the following: Given a program - considered as a process, we may ignore the particular notation used for writing its instructions - \( prg \in Prog \) and its set of data \( D=Input \cup Output \), a possible execution \( \text{exec}_i, i \in I, \) i.e., the program semantics, is viewed as a partially defined function \( f_i=\text{exec}_i(prg): D \to D, \) \( \text{dom}(f_i)=d_i \subseteq Input \). Since the model introduced in Manes and Arbib [1986] [13] is deterministic, different executions \( f_i \) of \( prg \) should have disjoint domains, an input determining at most one output. The semantics is defined as a sum of all possible executions corresponding to different data

\[
\text{sem}(prg)=\sum (f_i; i \in I) \in P=P\text{fn}(D,D),
\]

where \( P\text{fn}(D,D) \) is the set of partially defined functions \( D \to D \), the sum being defined in a convenient algebraic structure called partially additive monoid, with

\[
\text{dom}(f)=\cup (\text{dom}(f_i): i \in I), \text{dom}(f_j) \cap \text{dom}(f_k)=\emptyset, j \neq k,
\]

\[
f(x):=\text{if } (\exists i \in I) \text{ and } (x \in \text{dom}(f_i)) \text{ then } f_i(x) \text{ else undefined.}
\]

The reference structure is a partially additive naturally ordered or sum-ordered associative semiring \( (P\text{fn}(D,D), \sum, \cdot, 0, 1, \leq) \) (see Rudeanu and Vaida [2004][15], Vaida [2001] [18], [2005] [19]). The multiplication \( \cdot \) is a concatenation, a possible interpretation being the composition of functions corresponding to the sequential execution of instructions, \( \text{dom}(0)=\emptyset, 1=1_D \) and \( f \leq g \) means that \( g \) is an extension of \( f \) (the approximation ordering). Note that there is no way to bound two functions that have different values for at least one argument. \( P \) is a (additively)-cancellative, i.e., \( f+1=f+f_2 \) implies \( f_1=f_2 \).

The above structure for \( P\text{fn}(D,D) \) – the model we use when programs are studied as mathematical objects – should be further explained. For doing this we need concepts and terminology. (If you think that these...
coming elements are just imagination/fiction, you are wrong. For sure there is no perception of reality which is only physical. In any knowledge, the concepts, the intellectual components, constitute an indispensable raw material, true enough in many cases at a low level).

**Definition.** A naturally ordered/sum-ordered partial semiring is a partial algebra \((S, +, \cdot, 0, 1, \leq)\) with the following properties: \((S,+, 0)\) is a partial commutative monoid with a neutral element 0 called zero, the addition + being only partially defined (not as in Classical Algebra where operations are totally defined); this monoid is partially associative, in the sense that if one side of the associative rule exists, the other exists as well and their equality holds; \((S; +, 0, \leq)\) is partially ordered by \(a \leq b \iff \exists c \in S, \text{such that } b = a + c\) (in \(Pfn(D,D)\) equivalent to the approximation ordering previously defined, compatible with addition, in the sense that if \(b + c \in S\) exists as well and \(a + c \leq b + c; (S, \cdot, 1, 0, \leq)\) is a partially ordered monoid with an identity element 1 and an annihilator element 0; if the sum \(b+c\in S\) exists the following rules of partial distributivity are satisfied:

\[
a(b+c) = ab + ac \quad \text{and} \quad (b+c)a = ba + ca, \forall a \in S,
\]
in the sense that the sums of the right side exist as well and the equalities hold.

The structure \(P = (Pfn(D, D), 0, \leq)\) is not a lattice because it is not upper directed since \(f, g \leq h \) would imply \(f(x) = g(x) = h(x), \) for any \(x \in d_f \cap d_g\) \((d_f = \text{dom}(f), \text{similarly for } g).\) It is a lower semilattice with first element 0 by taking

\[
d_{f \wedge g} = \{x \in D \mid x \in d_f \cap d_g \text{ and } f(x) = g(x) \},
\]
\[
\inf (f, g) = f \wedge g \text{ and } (f \wedge g)(x) = f(x) = g(x), \forall x \in d_{f \wedge g}.
\]

Consequently, one has the Riesz Interpolation Property (RIP) if \(u, v \leq x, y \) then \(\exists z \in P \text{ “between” such that } z \text{ is an element of } U(u, v) \text{ and } L(x, y).\)

Moreover, \(P\) is a multilattice. Indeed, if for \(f\) and \(g\) there is a common upper bound \(\exists h \in Pfn(D,D)\) such that \(f, g \leq h\) then \(\exists \sup (f, g) = f \vee g = k \in Pfn (D,D).\) Indeed, we take

\[
d_{f \vee g} = d_f \cup d_g
\]
\[
(f \vee g)(x) = \text{ if } x \in d_f \text{ then } f(x) \text{ else } g(x), \ x \in d_{f \vee g}.
\]
Readily, \( d_f, d_g \leq d_{f \lor g} \). From \( f, g \leq h \), we obtain \( d_{f \lor g} \leq d_h \) and for \( x \in d_{f \lor g} \), if \( x \in d_f \) then \( (f \lor g)(x) = f(x) = h(x) \) since \( f \leq h \) and similarly for \( g \).

In fact \( P \) is a particular case of a multilattice called \textit{nearlattice} defined in Cornish and Noor [1980] [8].

The multilattice operation \( f \lor g \) can be seen as a partially defined algebraic sum if we extend the definition of summability from the family of disjoint functions to the family with overlapping functions, i.e., functions which coincide on their domain overlaps. The support structure would be a partially additive naturally ordered associative semiring \( P_1=(Pfn(D,D), \sum_1, \cdot , 0, 1, \leq) \).

In \( P_1 \) if the sum \( f +_1 g \) exists then it coincides with \( f \lor g \). This sum exists iff \( f \) and \( g \) are compatible, in the sense that there exists \( h \in Pfn(D,D) \) such that \( f, g \leq h \). \( P_1 \) is a (additively) idempotent.

Three more properties of \( Pfn(D,D) \) related to its structure can be provided merely for motivating a further study. The proofs should remain for another paper.

The Sum Decomposition Property (SDP)

given if \( 0 \leq x \leq a + b \) then there exists \( s, t \) such that \( 0 \leq s \leq a, 0 \leq t \leq b \) and \( x = s + t \).

is equivalent with (RIP) under rather general conditions but this equivalence is not proved for \( P \) and therefore an independent proof is needed.

**Property 1.** If \( f, g, h \in Pfn(D,D) \) and \( f \leq g +_1 h \) then \( g_1 \) and \( h_1 \) exist such that \( g_1 \leq g, h_1 \leq h \) and \( f = g_1 +_1 h, \forall x \in \text{dom}(f) \).

**Property 2.** Each directed family of functions, i.e., such that any pair of functions has an upper bound, not necessarily in the family, admits a \textit{sup}.
References


