Spectral Automorphisms in Quantum Logics

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Abstract In the first part of the article, we consider sharp quantum logics, represented by orthomodular lattices. We present an attempt to build an analogue of the Hilbert space spectral theory in abstract orthomodular lattices, using the notion of spectral automorphism. In the second part, we discuss unsharp quantum logics, represented by effect algebras. We generalize spectral automorphisms to effect algebras and obtain, in this framework, similar results to the ones in orthomodular lattices. In the last section, we present results concerning atomic compression base effect algebras.

The original results of the author (obtained in collaboration or alone) presented in this paper are part of his Ph.D. thesis [8] and have been published in [9, 11, 12, 37].

Keywords quantum logic · orthomodular poset · orthomodular lattice · effect algebra · spectral automorphism · spectral theory

Introduction

Quantum logics are logic-algebraic structures that arise in the study of the foundations of quantum mechanics.

According to the conventional Hilbert space formulation of quantum mechanics, states and observables are represented by operators in a Hilbert space associated to the quantum system under investigation, the so-called state space. Propositions, which represent yes-no experiments concerning the system, form an orthomodular lattice, isomorphic to the partially ordered set of projection operators $\mathcal{P}(H)$ on the state space $H$. It is called the logic associated to the quantum system. Thus, orthomodular lattices, which are sometimes assumed to be complete, atomic and to fulfill the covering property (just like $\mathcal{P}(H)$) bear the name of quantum logics. We propose and investigate the following question:

What amount of quantum mechanics is coded into the structure of the propositional system?

In other words, we intend to investigate to what extent some of the fundamental physical facts concerning quantum systems can be described in the more general framework of orthomodular lattices, without the support of Hilbert space-specific tools.

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With this question in mind, we attempt to build, in abstract orthomodular lattices, something similar to the spectral theory in Hilbert space. For this purpose, we introduce and study spectral automorphisms.

According to the contemporary theory of quantum measurement, yes-no measurements that may be unsharp, called effects, are represented by so-called effect operators, self-adjoint positive operators on the state space $H$, smaller than identity. As an abstraction of the structure of the set of effect operators, the effect algebra structure is defined.

In the second part of the article, we move our investigation to the framework of unsharp quantum logics, represented by effect algebras. We generalize spectral automorphisms to effect algebras and obtain in this framework results that are analogous to the ones obtained in orthomodular lattices.

Finally, as a rather separate undertaking, we study atomic effect algebras endowed with a family of morphisms called compression base, analyzing the consequences of atoms being foci of compressions in the compression base. We then apply some of the obtained results to the particular case of effect algebras endowed with a sequential product.

The results presented in this article are contained in author’s PhD thesis [8]. Author’s original results can be found in [9, 11, 12, 37].

The structure of this article is as follows:

**Section 1.** The first section is devoted to a presentation of orthomodular structures such as orthomodular posets and lattices and of their basic properties. The physically meaningful relation of compatibility is discussed. Blocks, commutants and center of such structures are covered. The last section of the section is dedicated to atomicity, as well as covering and exchange properties. The facts presented in this section are covered in various monographs, such as, e.g., [39, 47, 52, 54, 63].

**Section 2.** The second section contains a discussion of the problem concerning the possibility of embedding quantum logics into classical ones. The origin of this problem can be traced back to a famous paper of Einstein, Podolsky and Rosen, where authors conjectured that a “completion” of quantum mechanical formalism, leading to its “embedding” into a larger, classical and deterministic theory is possible.

We give an overview of classical and newer results concerning this matter by Kochen and Specker [40], Zierler and Schlessinger [65], Calude, Hertling and Svozil [6], Harding and Ptak [36] in a unitary treatment.

**Section 3.** This section consists of results concerning spectral automorphisms in orthomodular lattices. We introduce spectral automorphisms, define their spectra and study a few examples. Then, we analyze the possibility of constructing such automorphisms in products and horizontal sums of lattices. A factorization of the spectrum of a spectral automorphism is shown. We give various characterizations, as well as necessary or sufficient conditions for an automorphism to be spectral or for a Boolean algebra to be its spectrum. Then we discuss the consequences of the fact that a physical theory, represented by a quantum logic, admits spectral automorphisms.

The last part of this section addresses the problem of the unitary time evolution of a system from the point of view of the spectral automorphisms theory. An analogue of the Stone theorem concerning strongly continuous one-parameter unitary groups is given. The results in this section have been published in [37, 12].

**Section 4.** We present background information on unsharp quantum logics, as repre-
presented by effect algebras. We discuss special elements, coexistence relation, which generalizes compatibility from orthomodular posets, various substructures and important classes of effect algebras, as well as automorphisms in effect algebras. The facts presented in this section can be found, e.g., in the book of Dvurečenskij and Pulmannová [15] which gathers many of the recent results in the field of quantum structures.

**Section 5.** Sequential, compressible and compression base effect algebras, which will be used in the sequel, are discussed. They were introduced by Gudder [29, 30] and Gudder and Greechie [32].

Sequential product in effect algebras formalizes the case of sequentially performed measurements. The prototypical example of a sequential product is defined on the set $E(H)$ of effects operators by $A \circ B = A^{1/2}BA^{1/2}$.

The set $E(H)$ of effect operators can be endowed with a family of morphisms $(J_P)_{P \in \mathcal{P}(H)}$ defined by $J_P(A) = PAP$, called compressions and indexed by the projection operators $P \in \mathcal{P}(H)$ which are also called the foci of compressions. The family $(J_P)_{P \in \mathcal{P}(H)}$ is said to form a compression base of $E(H)$. Inspired by the main features of the family $(J_P)_{P \in \mathcal{P}(H)}$, the notions of compression, compression base and compressible effect algebra were introduced in abstract effect algebras. As it turns out, compression base effect algebras generalize sequential, as well as compressible effect algebras.

**Section 6.** In this section, we present a generalization of spectral automorphisms to compression base effect algebras, which are currently considered as the appropriate mathematical structures for representing physical systems [22]. We give characterizations of spectral automorphisms in compression base effect algebras and various properties of spectral automorphisms and of their spectra. In order to evaluate how well our theory performs in practice, we apply it to an example of a spectral automorphism on the standard effect algebra of a finite-dimensional Hilbert space and we show the consequences of spectrality of an automorphism for the unitary Hilbert space operator that generates it. In the last section, spectral families of automorphisms are discussed and an effect algebra version of the Stone-type theorem in Section 3 is obtained. The results of this section have been published in [9].

**Section 7.** In the last section, we discuss atomic compression base effect algebras and the consequences of atoms being foci of compressions. Part of this work generalizes results obtained by Tkadlec [60] in atomic sequential effect algebras. The notion of projection-atomicity is introduced and studied and conditions that force a compression base effect algebra or the set of compression foci to be Boolean are given. We apply some of these results to the important particular case of sequential effect algebra and present strengthened versions of previous results obtained by Gudder and Greechie [32] and Tkadlec [60]. The results presented here have been published in [11].

**Preliminaries: sharp quantum logics**

Finding a mathematical and logical model for quantum mechanics has been a challenge since the first part of the twentieth century. Researchers like von Neumann, Birkhoff, Husimi, Dirac, Mackey, Piron and many others have contributed to this task. As a result of their work, Hilbert space theory has been established as the appropriate mathematical framework for the study of quantum mechanics. At the same time, as it was clear that the facts concerning the measurements of complementary variable in quantum mechanics, such as, e.g., position and momentum, are inconsistent with the distributivity law that works within classical Boolean logic, a new type of logic became necessary. With their historical paper “The Logic of Quantum Mechanics” (1936), Garrett Birkhoff and John von Neumann started the
search for a quantum logic.

In what follows, we will try to briefly explain the meaning of the essential elements of the axiomatic model of such a quantum logic. In the meantime, this will provide us with the physical interpretation of the results obtained in the framework of this model. Detailed descriptions of the way that this axiomatic model arises can be found in, e.g., [2, 46, 52, 63].

The mathematical modeling in quantum physics often has a speculative character, sometimes involving philosophical problems of physics. After this short introduction, we will devote ourselves to the study of the mathematical theory of quantum logics. Nevertheless, we hope that our results remain meaningful from a physical point of view.

The standard Hilbert space formulation of quantum mechanics is based on a few essential notions such as states, observables, propositions pertaining to a quantum system under investigation. Following mainly Mackey’s approach [46] and its discussion by Beltrametti and Cassinelli in [2], let us try to sketch an explanation of their meaning, mathematical models and mutual relations.

By a yes-no experiment, we mean a test performed on a physical system using a measuring apparatus that has only two possible outcomes, which can be labeled “yes” and “no”. By preparation procedure of the physical system we understand all the information about the operations performed on the system until the moment when the test is performed.

We consider two preparation procedures of a physical system to be equivalent if we cannot distinguish them by any yes-no experiment (i.e., every yes-no experiment has the same probability of a “yes” outcome for both preparations). We shall call a state of the system an equivalence class of preparation procedures. Let us denote by $\mathcal{S}$ the set of states associated to a physical system.

An observable of a physical system is generally understood as referring to a measurable physical quantity of the system. It is assumed that such a physical quantity denoted by $A$ takes values on the real axis. If $M$ is a Borel subset of $\mathbb{R}$, then the question whether the measured value of $A$ lies in $M$ is a yes-no experiment that we denote by $(A, M)$. We consider two such yes-no experiments $(A, M)$ and $(B, N)$ to be equivalent if they have the same probability of a “yes” outcome in every possible state of the system. An equivalence class of yes-no experiments of the above type will be called a proposition about the system. Let us denote by $\mathcal{L}$ the set of propositions associated to a physical system.

Let us remark that every state $s \in \mathcal{S}$ of the system defines—and can be regarded as being described by—a probability function defined on the set $\mathcal{L}$, taking values in the interval $[0,1]$ which associates to every proposition $p \in \mathcal{L}$ the probability $s(p)$ of a “yes” outcome when the system is prepared in the state $s$.

It is now intuitively clear that an observable $A$ of a system in a state $s$ can be completely described by the knowledge of the probability $s(p)$ for all propositions $p$ represented by yes-no experiments $(A, M)$, with $M$ spanning the Borel sets of $\mathbb{R}$ (i.e., by the knowledge, for all Borel sets $M$, of the probability that the measuring of $A$ in the state $s$ will yield a result in $M$).

A natural ordering is defined on the set $\mathcal{L}$ of propositions by putting $p \leq q$ if and only if $s(p) \leq s(q)$ for all $s \in \mathcal{S}$. Let us remark that the yes-no experiment $(A, \emptyset)$ is a representative of the trivial proposition that has always (i.e., in all states) the answer “no”, which we will denote by $0$ and the yes-no experiment $(A, \mathbb{R})$ is a representative of the trivial proposition that has always the answer “yes”, which we will denote by $1$. Then $0 \leq p \leq 1$ for all $p \in \mathcal{L}$.

Orthogonality can be defined on $\mathcal{L}$ as follows: we call propositions $p$ and $q$ orthogonal (in symbols, $p \perp q$) if $s(p) + s(q) \leq 1$ for all states $s \in \mathcal{S}$.

To every proposition $p$, represented by $(A, M)$, we can associate its negation, represented by $(A, \mathbb{R} \setminus M)$ and denoted by $p'$.

According to a crucial axiom of Mackey [46], for every sequence of pairwise orthogonal
propositions $p_1, p_2, \ldots \in \mathcal{L}$, there exist a proposition $q \in \mathcal{L}$ such that $s(q) + s(p_1) + s(p_2) + \ldots = 1$ for all $s \in S$.

Anticipating some terminology (for which the reader may consult section 1.1), we can say that, as a consequence of this axiom, the mapping $p \mapsto p'$ becomes an orthocomplementation and $\mathcal{L}$ becomes an orthomodular lattice which is also $\sigma$-complete. Moreover, the probability function $s : \mathcal{L} \to [0, 1]$ associated to every state $s \in S$ becomes a probability measure on $\mathcal{L}$, in the sense of the following definition:

If $\mathcal{L}$ is a $\sigma$-complete orthomodular lattice, a mapping $s : \mathcal{L} \to [0, 1]$ is a probability measure on $\mathcal{L}$ if $s(1) = 1$ and for every sequence $(p_n)_{n \in \mathbb{N}}$ of pairwise orthogonal elements in $\mathcal{L}$, $s(\bigvee_{n \in \mathbb{N}} p_n) = \sum_{n \in \mathbb{N}} s(p_n)$.

We are led to the idea that the states of a physical system can be identified with probability measures on the orthomodular lattice of propositions of the system. Let us note, as a side remark, that sometimes, the lattice of the propositions of a system is not assumed to be $\sigma$-complete and the states (probability measures) are not assumed to be $\sigma$-additive as in the above definition, but only additive, as in Definition 2.2.3.

In his celebrated book, “Mathematical Foundations of Quantum Mechanics” [49], John von Neumann introduced Hilbert space as the appropriate mathematical framework for quantum mechanics. According to the conventional Hilbert space formulation of quantum mechanics, observables are represented by self-adjoint operators on the Hilbert space $H$ associated to the quantum system, while states are represented by density operators (i.e., trace class operators of trace 1) on $H$. At this point, one might ask one’s self what is the connection between these mathematical representations and the notions of state and observable previously described here. To answer this question, an additional assumption is necessary, and this assumption is precisely the content of Mackey’s “quantum” axiom VII. This axiom states that the partially ordered set of all propositions in quantum mechanics is isomorphic to the partially ordered set of projection operators (or, equivalently, of closed subspaces) of a separable, infinite dimensional, complex Hilbert space.

Let us recall that projection operators on a Hilbert space $H$ are its self-adjoint idempotents. To each projection operator there corresponds a unique closed linear subspace of the Hilbert space $H$ which is the range of the projection. This is a one-to-one correspondence. The set of projection operators on a Hilbert space $H$, which we will denote henceforward by $\mathcal{P}(H)$ is organized as a complete orthomodular lattice which, by virtue of the above said correspondence, is isomorphical to (and can be identified with) the complete orthomodular lattice of closed subspaces of $H$ (see, e.g., [35, 39]). To be precise, if $P_1, P_2 \in \mathcal{P}(H)$ and $M_1, M_2$ are their respective ranges, the order relation in $\mathcal{P}(H)$ is defined by putting $P_1 \leq P_2$ whenever $M_1 \subseteq M_2$. The orthocomplementation is defined on $\mathcal{P}(H)$ by $P' = 1 - P$ (where 1 denotes the identity of $H$), i.e., $P'$ is the projection onto the closed subspace that is orthogonal to the range of $P$.

Let us return to Mackey’s axiom VII which we also called the “quantum” axiom. This name is a reference to the fact that it is precisely the assumption of this axiom that makes the difference between quantum systems and other physical systems. We should remark that Mackey himself commented [46] that this axiom seems “entirely ad hoc” and that “we are far from being forced to accept this axiom as logically inevitable”. He adds, however, that “we make it because it ‘works’, that is, it leads to a theory which explains physical phenomena and successfully predicts the results of experiments”.

By virtue of Mackey’s axiom VII, we can identify the lattice of propositions $\mathcal{L}$ of a quantum system and the lattice $\mathcal{P}(H)$ of projectors on the Hilbert space associated to the quantum system. A state, represented by a density operator $\rho$, induces a probability measure on $\mathcal{P}(H)$ by $P \mapsto \text{tr}(\rho P)$ for all $P \in \mathcal{P}(H)$. Conversely, according to a famous and highly nontrivial theorem of Gleason (see, e.g., [63, 2]), if $\text{dim}(H) \geq 3$, then every probability
measure on $\mathcal{P}(H)$ arises from a density operator $\rho$ on $H$ by $P \mapsto \text{tr}(\rho P)$, for all $P \in \mathcal{P}(H)$. Therefore, the set of density operators and the set of probability measures on $\mathcal{P}(H)$ are in a one-to-one correspondence which, moreover, preserves the convex structure of both sets. It should be mentioned here that the extreme points of the convex set of states are called pure states and they correspond to density operators which are projection operators on unidimensional subspaces—so-called rays—of $H$. By a slight abuse of language, the unit vectors generating the rays are called sometimes pure states.

We have sketched, up to this point, the explanation of the correlation between states seen as equivalence classes of preparation procedures, probability measures on the orthomodular lattice of propositions (or projection operators) and density operators. Let us try to accomplish the corresponding task concerning observables. This will also allow us to add a final missing step to our explanation on states.

According to von Neumann, an observable $A$ is represented by a self-adjoint operator (which we will denote also by $A$) on the Hilbert space $H$ associated to the system. The spectral values of the operator are interpreted as the possible outcome of the measurement of the observable. According to the spectral theorem for self-adjoint operators (see, e.g., [50, 55]), to $A$ corresponds a spectral projection valued measure (PV-measure), i.e. a mapping $M \mapsto P_A(M)$ which associates to every Borel set $M$ of $\mathbb{R}$ a projection operator $P_A(M)$, such that:

- $P_A(\emptyset) = 0$, $P_A(\mathbb{R}) = 1$;
- $P_A(\bigcup_{n \in \mathbb{N}} M_n) = \sum_{n \in \mathbb{N}} P_A(M_n)$ for every sequence of mutually disjoint Borel sets $(M_n)_{n \in \mathbb{N}}$ (the series on the right converges in the strong operator topology);
- $P_A(M_1 \cap M_2) = P_A(M_1)P_A(M_2)$.

By Mackey’s axiom $\text{VII}$, as mentioned before, the set $\mathcal{L}$ of propositions can be identified with the set of projection operators $\mathcal{P}(H)$. It is natural to consider that the proposition $p \in \mathcal{L}$ represented by the yes-no experiment $(A, M)$ (i.e., by the question whether the measured value of $A$ lies in the Borel set $M$) corresponds to the projection operator $P_A(M)$. Moreover, the probability of obtaining for the observable $A$ a measured value within the Borel set $M$, when the system is prepared in a state $s$ represented by the density operator $\rho$ is given by $\text{tr}(\rho P_A(M))$. It is not difficult to see that the mapping $M \mapsto \text{tr}(\rho P_A(M))$ defined on the Borel sets of $\mathbb{R}$ is a probability measure.

Summarizing, we have justified the correspondence between observables as self-adjoint operators and observables as PV-measures. Moreover, we have specified in what way, for an observable defined as a self-adjoint operator $A$, we can compute the probability of obtaining a measured value within a Borel set $M$, when the system is prepared in a state $s$. As we mentioned before, when we introduced the notion of observable, this intuitively corresponds to a complete description of the observable.

Finally, let us remark, as an argument supporting Mackey’s axiom $\text{VII}$, that if the spectrum of an observable represents the possible outcomes of its measurements, the spectrum of projection operators is a subset of $\{0, 1\}$, which suggests the interpretation of observable represented by projection operators as propositions corresponding to yes-no experiments.

In view of the foregoing description of the mathematical model of a quantum system, it should be clear why the partially ordered set of propositions of the system is called the logic of the system and hence, why orthomodular lattices (which are sometimes assumed to be complete, or to have the covering property, like the lattice $\mathcal{P}(H)$ which they generalize) are called quantum logics.

An important question that one might ask is formulated by Beltrametti and Cassinelli [2] as follows:

“\text{The fact that states and physical quantities can be defined in terms of } \mathcal{P}(H) \text{ suggests the}..."
following problem: Take from the outset a partially ordered set \( L \), to be physically interpreted as the set of ‘propositions’ and having some of the properties of \( \mathcal{P}(H) \) but without any notion of Hilbert space. Consider the set \( S \) of all probability measures on \( L \) and the set \( \mathcal{O} \) of all functions from \( \mathcal{B}(\mathbb{R}) \) into \( L \) that have the formal properties of spectral measures. Then, is it possible to determine a Hilbert space \( H \) such that \( L \) is identified with \( \mathcal{P}(H) \), \( S \) with the set of all density operators on \( H \) and \( \mathcal{O} \) with the set of all self-adjoint operators on \( H \) ? Briefly, the question is: to what extent is the Hilbert space description of quantum systems coded into the ordered structure of propositions?”

A great part of the work presented here, is essentially motivated by another—in a way, complementary—question: is it possible to avoid the use of Hilbert space specific tools and replace them with instruments belonging to the lattice of propositions, in the description of quantum systems?

1 Basics on orthomodular structures

In this introductive section we present the main orthomodular structures which arise from quantum mechanics—most notably, orthomodular posets and lattices—and their properties. The different facts presented in this section are covered in various monographs, like, e.g., [2, 39, 47, 50, 52, 54, 63].

1.1 Definitions of orthomodular structures

**Definition 1.1.1** Let \((P, \leq)\) be a bounded poset. A unary operation ‘ on \( P \) such that, for every \( a, b \in P \), the following conditions are fulfilled:

1. \( a \leq b \) implies \( b' \leq a' \),
2. \( a'' = a \),
3. \( a \lor a' = 1 \) and \( a \land a' = 0 \),

is an orthocomplementation on \( P \).

**Definition 1.1.2** A bounded poset with an orthocomplementation is an orthoposet. An orthoposet which is a lattice is an ortholattice.

**Definition 1.1.3** A relation orthogonal, denoted by “\( \perp \)” is defined for elements \( a, b \) of an orthoposet by

\[ a \perp b \iff a \leq b' \]

**Definition 1.1.4** An orthoposet (ortholattice) \((P, \leq, ')\) satisfies the orthomodular law if for every \( a, b \in P \),

\[ a \leq b \text{ implies there exists } c \in P, c \perp a \text{ such that } b = a \lor c \] (OM1)

**Definition 1.1.5** An orthoposet with the property that every pair of orthogonal elements has supremum and satisfying the orthomodular law is an orthomodular poset. If, moreover, the supremum exists for every countable set of pairwise orthogonal elements, it is a \( \sigma \)-complete orthomodular poset.

**Definition 1.1.6** An ortholattice satisfying the orthomodular law is an orthomodular lattice.
1.2 Compatibility. Basic properties

Compatible pairs represent simultaneously verifiable events, hence their importance in the axiomatics of quantum theories.

**Definition 1.2.1** Let $P$ be an orthomodular poset. Elements $a, b \in P$ are compatible (in $P$) if there exist mutually orthogonal elements $a_1, b_1, c \in P$ such that $a = a_1 \lor c$ and $b = b_1 \lor c$. In this case we will write $a \leftrightarrow_P b$, or just $a \leftrightarrow b$, when there’s no risk of confusion. For $M$ a subset of $P$, we shall write $a \leftrightarrow M$ when $a \leftrightarrow m$ for every $m \in M$.

**Lemma 1.2.2** Let $a$ and $b$ be elements of an orthomodular poset $P$. Then:

1. $a \leq b$ implies $a \leftrightarrow b$;
2. $a \perp b$ if and only if $a \leftrightarrow b$ and $a \land b = 0$;
3. the following are equivalent: $a \leftrightarrow b$, $a \leftrightarrow b'$, $a' \leftrightarrow b'$.

**Theorem 1.2.3** [39, Ch. 1, Section 3, Proposition 4] Let $(L, \leq', \lor, \land, \lnot')$ be an orthomodular lattice and $M$ be a subset such that $\bigvee M$ exists. If $b \in L$ is such that $b \leftrightarrow M$, then:

1. $b \leftrightarrow \bigvee M$
2. $b \land (\bigvee M) = \bigvee \{b \land m : m \in M\}$

**Proposition 1.2.4** [54, Proposition 1.3.11] Let $(L, \leq', \lor, \land, \lnot')$ be an orthomodular lattice and $a, b, c \in L$. If $a \leftrightarrow b$ and $a \leftrightarrow c$, then $\{a, b, c\}$ is a distributive triple.

**Corollary 1.2.5** An orthomodular poset is a Boolean algebra if and only if every pair of its elements is compatible.

1.3 Orthomodular substructures

**Definition 1.3.1** A subset of an ortholattice is a subalgebra if it contains the least and greatest elements and it is closed under lattice operations $\lor, \land$ and orthocomplementation $\lnot'$.

**Definition 1.3.2** Let $L$ be an ortholattice. A subalgebra of $L$ which is a Boolean algebra with the induced operations from $L$ is a Boolean subalgebra of $L$.

**Definition 1.3.3** The maximal Boolean subalgebras of an orthomodular lattice are called its blocks.

**Definition 1.3.4** A subset of an orthomodular poset is a suborthoposet if it contains the least and greatest elements and it is closed under orthocomplementation and under suprema of orthogonal pairs.

**Definition 1.3.5** A suborthoposet of an orthomodular poset $P$ which is a Boolean algebra with the induced from $P$ order, orthocomplementation and lattice operations, is called a Boolean subalgebra of the orthomodular poset $P$.

If $P$ is an orthomodular lattice, this notion of Boolean subalgebra coincides with the one given in Definition 1.3.2.
Definition 1.3.6  Let $P$ be an orthomodular poset. An element $a \in P$ is central if it is compatible with every other element of $P$. The set of central elements of $P$ is the center of $P$, denoted henceforth by $\tilde{C}(P)$.

Proposition 1.3.7  The center of an orthomodular poset $P$ is a Boolean subalgebra of $P$.

Definition 1.3.8  Let $P$ be an orthomodular poset and $\Delta \subseteq P$. The commutant of $\Delta$ in $P$ is the set $\{a \in P: a \leftrightarrow \Delta\}$. It will be denoted henceforth by $K_P(\Delta)$ or, if there is no possibility of confusion about $P$, simply by $K(\Delta)$.

1.4 Atomicity. Covering and exchange properties. Lattices with dimension

Definition 1.4.1  Let $L$ be a poset with least element $0$. A minimal non-zero element of $L$ is an atom. $L$ is atomic if every its element dominates (at least) an atom of $L$. It is atomistic if every element is the supremum of the atoms it dominates. Let $\Omega_a$ denote the set of atoms dominated by an element $a \in L$, and $\Omega(L)$ denote the set of atoms of $L$.

Proposition 1.4.2  Every atomic orthomodular lattice is atomistic.

Proposition 1.4.3  If $B$ is a Boolean subalgebra of the orthomodular lattice $L$, $a$ is an atom of $B$ and $\omega \in L$ such that $\omega \leq a$, then $\omega \leftrightarrow B$.

Proposition 1.4.4  Let $L$ be an atomic orthomodular lattice. For every element $a \in L$, there exists a maximal family $\{\alpha_i\}_{i \in I}$ of mutually orthogonal atoms in $\Omega_a$. Then, $a = \bigvee_{i \in I} \alpha_i$.

Definition 1.4.5  Let $L$ be an atomic orthomodular lattice and $a \in L$. A maximal family $\{\alpha_i\}_{i \in I}$ of mutually orthogonal atoms in $\Omega_a$ is a basis of $a$. A basis of $1 \in L$ is also called a basis of the lattice $L$.

Theorem 1.4.6  [see [32, Ch.1, Section 4, Lemma 2]] Let $L$ be an orthomodular lattice and $B$ a Boolean subalgebra of $L$. If $B$ is a block of $L$ then the atoms of $B$ are atoms of $L$. Conversely, if $B$ is atomic and its atoms are atoms of $L$, it is a block of $L$.

1.5 Morphisms in orthomodular structures

Definition 1.5.1  Let $L_1, L_2$ be orthomodular posets. A mapping $h : L_1 \rightarrow L_2$ is a morphism of orthomodular posets if the following conditions are satisfied:

(1) $h(1) = 1$;
(2) $a \perp b$ implies $h(a) \perp h(b)$, for every $a, b \in L_1$;
(3) $h(a \lor b) = h(a) \lor h(b)$, for every pair of orthogonal elements $a, b \in L_1$.

Definition 1.5.2  Let $L_1, L_2$ be orthomodular lattices (Boolean algebras). A mapping $h : L_1 \rightarrow L_2$ is a morphism of orthomodular lattices (Boolean algebras, respectively) if it is a morphism of orthomodular posets and it preserves the join of arbitrary pairs of elements.

Definition 1.5.3  A morphism $h : L_1 \rightarrow L_2$ of orthomodular posets (or orthomodular lattices, or Boolean algebras, respectively) is an embedding if, for every $a, b \in L_1$, $h(a) \perp h(b)$ implies $a \perp b$. It is an isomorphism if it is bijective and its inverse $h^{-1} : L_2 \rightarrow L_1$ is also a morphism. An isomorphism $h : L \rightarrow L$ is an automorphism of $L$. 

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Proposition 1.5.4 Let $L$ be an orthomodular poset (or an orthomodular lattice, or a Boolean algebra). A mapping $h : L \to L$ is an automorphism if and only if it satisfies the following conditions:

1. $h(1) = 1$;
2. $a \perp b$ implies $h(a) \perp h(b)$, for all $a, b \in L$;
3. $a \leq b$ if and only if $h(a) \leq h(b)$, for all $a, b \in L$;
4. $h$ is surjective.

The following result shows that every automorphism of an atomic complete orthomodular lattice is uniquely determined by its restriction to the set of atoms, which is a bijective map that preserves orthogonality both ways.

Theorem 1.5.5 \cite[Theorem 2.46]{52} Let $L$ be an atomic complete orthomodular lattice. For every automorphism $h$ of $L$, its restriction to $\Omega(L)$ is $\chi : \Omega(L) \to \Omega(L)$ satisfying the conditions:

1. $\chi$ bijective;
2. $\alpha \perp \beta$ if and only if $\chi(\alpha) \perp \chi(\beta)$, for all $\alpha, \beta \in \Omega(L)$.

Conversely, for every mapping $\chi : \Omega(L) \to \Omega(L)$ satisfying the above conditions (1), (2) there exists a unique automorphism $h$ of $L$ such that its restriction to $\Omega(L)$ is $\chi$.

2 Understanding the logic of quantum mechanics in classical terms

The problem of embedding quantum logics into classical ones is very old. Its origin can be traced back to a well known article of Einstein, Podolsky and Rosen (EPR) \cite{16}. In this historic paper, the authors conjectured that a “completion” of quantum mechanical formalism, leading to its “embedding” into a larger, classical and deterministic theory (from the algebraic and logic point of view) is possible.

We explore various possibilities to embed quantum logics into classical ones. We discuss the different approaches and results obtained concerning this matter by e.g., Kochen and Specker \cite{40}, Zierler and Schlessinger \cite{65}, Calude, Hertling and Svozil \cite{6}, Harding and Ptak \cite{36}, thus offering an overview of what can be achieved in terms of classical understanding of quantum mechanics.

2.1 The impossibility of embedding a quantum logic into a classical one

Assuming that a “proper” embedding of a non-Boolean orthomodular lattice (a quantum logic) into a Boolean algebra (a classical logic) would exist, it would preserve compatibility. Since every pair of elements is compatible in a Boolean algebra, the same would have to be true in the orthomodular lattice, which contradicts to our assumption.

In what follows, we try to weaken the notion of embedding, in order to make it possible for an orthomodular lattice to be embedded (in this weaker sense) into a Boolean algebra. A natural idea to overcome the above mentioned contradiction is to only ask that an embedding preserves the join for orthogonal elements. In such a case, we can as well generalize our discussion to orthomodular posets instead of orthomodular lattices.
2.2 A characterization of orthomodular posets that can be embedded into Boolean algebras

Before stating the main result of this section, let us introduce a few important notions.

**Definition 2.2.1** An orthomodular poset \((P, \subseteq, c, \emptyset, X)\), where \(X\) is a nonempty set, \(P \subseteq 2^X\), order is defined by set-theoretical inclusion, the orthocomplement of an element \(A \in P\) is the set-theoretical complement of \(A\) relative to \(X\) (denoted by \(A^c\)) and \(\emptyset\) and \(X\) are the least and greatest elements of \(P\), respectively, is a set orthomodular poset.

**Definition 2.2.2** A set orthomodular poset with the property that, for every \(A, B \in P\), \(A \cup B \in P\) whenever \(A \cap B = \emptyset\) is a concrete orthomodular poset.

**Definition 2.2.3** Let \(P\) be an orthomodular poset. A mapping \(s : P \to [0, 1]\) such that:

1. \(s(1) = 1\);
2. \(s(a \lor b) = s(a) + s(b)\), whenever \(a, b \in P, a \perp b\)

is a state on \(P\). If, moreover, the range of \(s\) is \(\{0, 1\}\), then \(s\) is a two-valued state on \(P\).

**Definition 2.2.4** A set \(S\) of states on an orthomodular poset \(P\) is full (or order determining) if, for every \(a, b \in P\) with \(a \not\leq b\), there exists a state \(s \in S\) such that \(s(a) \not\leq s(b)\) (i.e., \(s(a) \leq s(b)\) for all \(s \in S\) implies \(a \leq b\)).

**Theorem 2.2.5** [see [24, Theorem 2.2.1] or [28]] An orthomodular poset has a representation as a concrete orthomodular poset if and only if it has a full set of two-valued states.

Let us remark that an orthomodular poset has a concrete representation if and only if it can be embedded into a Boolean algebra and therefore, Theorem 2.2.5 gives in fact a necessary and sufficient condition for the existence of such an embedding—the existence of a full set of two-valued states defined on the orthomodular poset. However, this condition is quite restrictive. For instance, the standard Hilbert space orthomodular lattice of projectors, for a Hilbert space \(H\) of dimension higher than 2, has no two-valued states, not to mention a full set.

We must conclude that in general, we cannot embed orthomodular posets or lattices into Boolean algebras, if we expect that such an embedding to preserve the join of orthogonal elements.

2.3 The Zierler-Schlessinger theory

Let us further weaken the embedding notion, asking the join to be preserved only for central elements.

**Definition 2.3.1** Let \(L\) be an orthomodular lattice and \(B\) a Boolean algebra. A mapping \(h : L \to B\) is a \(Z\)-embedding of \(L\) into \(B\) if the following conditions are satisfied:

1. \(h(1) = 1\);
2. \(h(a^c) = h(a)^c\) for every \(a \in L\);
3. \(a \perp b\) if and only if \(h(a) \perp h(b)\), for every \(a, b \in L\);
4. \(h(a \lor b) = h(a) \lor h(b)\), for every pair of central elements \(a, b \in L\).
Theorem 2.3.2  \cite{63} For every orthomodular lattice \( L \), there exist a \( Z \)-embedding into a power set Boolean algebra.

In many cases, the center of an orthomodular lattice is rather “poor” or even trivial. In these cases, the above theorem gives us a type of embedding that preserves the join of a very limited number of elements.

2.4 The result of Harding and Pták

The following result of J. Harding and P. Pták \cite{36} substantially improves the properties that can be obtained for an embedding of an orthomodular lattice into a Boolean algebra.

Theorem 2.4.1  \cite{36} Let \( L \) be an orthomodular lattice and let \( B \) be a Boolean subalgebra of \( L \). There exist a set \( S \) and a mapping \( h : L \rightarrow 2^S \) such that:

1. \( h(1) = S; \)
2. \( h(a') = h(a)^c; \)
3. \( a \perp b \) if and only if \( h(a) \cap h(b) = \emptyset; \)
4. \( h(a \lor b) = h(a) \cup h(b), \) for every \( a, b \in B; \)
5. \( h(a \lor b) = h(a) \cup h(b) \) whenever \( a \in \hat{C}(L). \)

3 Spectral automorphisms in orthomodular lattices

In this section we present original research results that were published in articles \cite{37} and \cite{12}. We develop the theory of spectral automorphisms in orthomodular lattices and obtain in this framework some results that are analogues of the ones in the spectral theory in Hilbert spaces.

3.1 Spectral automorphisms—the idea

Let \( H \) be a Hilbert space and \( \mathcal{P}(H) \) the orthomodular lattice of projection operators on \( H \). Automorphisms of \( \mathcal{P}(H) \) are of the form \( \varphi_U : \mathcal{P}(H) \rightarrow \mathcal{P}(H), \varphi_U(P) = UPU^{-1}, \) with \( U \) being a unitary or an antiunitary operator on \( H \). Let \( U \) be unitary and let \( B_U \) be the Boolean subalgebra of \( \mathcal{P}(H) \) that is the range of the spectral measure associated to \( U \). Then \( P \in \mathcal{P}(H) \) is \( \varphi_U \)-invariant if and only if \( UP = PU \) if and only if \( P \) commutes with \( B_U \) (i.e., commutes with every projection operator in \( B_U \)) if and only if \( P \leftrightarrow B_U \). This suggests the definition of spectral automorphisms in orthomodular lattices.

3.2 Definition and basic facts

Definition 3.2.1 Let \( L \) be an orthomodular lattice and \( \varphi \) be an automorphism of \( L \). The automorphism \( \varphi \) is spectral if there is a Boolean subalgebra \( B \) of \( L \) such that

\[ \varphi(a) = a \text{ if and only if } a \leftrightarrow B. \]  \hspace{1cm} (P1)

A Boolean subalgebra of \( L \) satisfying condition \((P1)\) is a spectral algebra of \( \varphi \). The set of \( \varphi \)-invariant elements of \( L \) is denoted by \( L_\varphi \).

Proposition 3.2.2 For any spectral automorphism, there exists the greatest Boolean subalgebra having the property \((P1)\)
Definition 3.2.3 If $\varphi : L \to L$ is a spectral automorphism, the greatest Boolean subalgebra having the property (P1) is called the spectrum of $\varphi$ and will be denoted by $\sigma_\varphi$.

Corollary 3.2.4 If an orthomodular lattice has a nontrivial spectral automorphism, then it cannot be Boolean.

Proposition 3.2.5 The automorphism $\varphi : L \to L$ is spectral if and only if there is a Boolean subalgebra $B$ of $L$ such that $L_\varphi = K(B)$. In this case, $\sigma_\varphi = \tilde{C}(L_\varphi)$.

Corollary 3.2.6 The automorphism $\varphi : L \to L$ is spectral if and only if $K(\tilde{C}(L_\varphi)) \subseteq L_\varphi$.

Example 3.2.7 Let $H$ be an $n$-dimensional complex Hilbert space and $\mathcal{P}(H)$ be the set of its projection operators. Let $Q$ be a 1-dimensional projection on $H$ and $Q'$ be its orthogonal complement. We define $U : H \to H$ as the symmetry of $H$ with respect to the hyperplane corresponding to $Q'$. It is easy to see that $U$ is a unitary operator, therefore $\varphi : \mathcal{P}(H) \to \mathcal{P}(H)$ defined by $\varphi(P) = UPU^{-1}$ is an automorphism of $\mathcal{P}(H)$. $B = \{0, Q, Q', 1\}$ is a Boolean subalgebra of $\mathcal{P}(H)$ fulfilling condition (P1) in Definition 3.2.1. Indeed, the set of $\varphi$-invariant elements, as well as the set of elements that are compatible with $B$, is $\mathcal{P}_0 \cup \mathcal{P}_0'$, where $\mathcal{P}_0 = \{A \in \mathcal{P}(H) : A \leq Q'\}$ and $\mathcal{P}_0'$ denotes the set of orthocomplements of the elements of $\mathcal{P}_0$.

3.3 Spectral automorphisms in products and horizontal sums

We discuss the construction of spectral automorphisms in products and horizontal sums of orthomodular lattices. A factorization of the spectra of spectral automorphisms is also studied.

Theorem 3.3.1 Let $L$ be the product of a collection $(L_i)_{i \in I}$ of orthomodular lattices and, for every $i \in I$, $\varphi_i$ be an automorphism of $L_i$. Let us define the mapping $\varphi : L \to L$ by $\varphi((a_i)_{i \in I}) = (\varphi_i(a_i))_{i \in I}$. Then:

1. $\varphi$ is an automorphism of $L$;
2. $\varphi$ is spectral if and only if $\varphi_i$ is spectral for every $i \in I$; in this case, $\sigma_\varphi = \prod_{i \in I} \sigma_{\varphi_i}$.

Lemma 3.3.2 Let $L$ be the horizontal sum of a collection $(L_i)_{i \in I}$ of orthomodular lattices such that every summand is minimal, $\varphi$ be an automorphism of $L$ such that $\varphi(L_i) \cap L_i \neq \{0, 1\}$ for some $i \in I$. Then the restriction $\varphi_i$ of $\varphi$ to $L_i$ is an automorphism of $L_i$.

Theorem 3.3.3 Let $L$ be the horizontal sum of a collection $(L_i)_{i \in I}$ of orthomodular lattices such that every summand is minimal and $\varphi$ be an automorphism of $L$.

1. If $\varphi$ is spectral then there is an $i \in I$ such that $L_\varphi \subseteq L_i$, $\sigma_\varphi \subseteq L_i$ and the restriction $\varphi_i$ of $\varphi$ to $L_i$ is a spectral automorphism of $L_i$ with $L_{\varphi_i} = L_\varphi$ and $\sigma_{\varphi_i} = \sigma_\varphi$.
2. If $L_\varphi \neq \{0, 1\}$ and there is an $i \in I$ such that $L_\varphi \subseteq L_i$ and the restriction $\varphi_i$ of $\varphi$ to $L_i$ is spectral then $\varphi$ is a spectral automorphism of $L$ and $L_\varphi = L_{\varphi_i}$, $\sigma_\varphi = \sigma_{\varphi_i}$.

Theorem 3.3.4 Let $L$ be an orthomodular lattice, $\varphi$ be a spectral automorphism of $L$ and $a \in L \setminus \{0, 1\}$ be $\varphi$-invariant. Let us denote by $\varphi_a$ the restriction of $\varphi$ to $[0, a]$, and let $B_\varphi = x \wedge \sigma_\varphi = \{x \wedge b : b \in \sigma_\varphi\}$ for every $x \in L$. Then:

1. $\varphi_a$ is a spectral automorphism of $[0, a]$ and $B_\varphi$ is its spectral algebra;
2. if $a \in \sigma_\varphi$, then $\sigma_{\varphi_a} = B_a$.
3. \( \sigma \varphi \) is isomorphic to the product \( B_a \times B_{a'} \).

**Question 3.3.5** Is it possible to omit the condition \( a \in \sigma \varphi \) in Theorem 3.3.4 (2)?

### 3.4 Characterizations of spectral automorphisms

**Theorem 3.4.1** Let \( L \) be an orthomodular lattice. An automorphism \( \varphi \) of \( L \) is spectral if and only if \( a \land b \in L \varphi \) for every \( a \in \tilde{C}(L \varphi) \) and \( b \in K(\tilde{C}(L \varphi)) \).

**Definition 3.4.2** Let \( L \) be an orthomodular lattice and \( \varphi \) an automorphism of \( L \). An element \( a \in L \) is totally \( \varphi \)-invariant if \( \varphi(b) = b \) for every \( b \in L \) with \( b \leq a \).

**Theorem 3.4.3** Let \( L \) be a complete orthomodular lattice and \( \varphi \) be an automorphism of \( L \) such that \( \tilde{C}(L \varphi) \) is atomic. Then \( \varphi \) is spectral if and only if all atoms of \( \tilde{C}(L \varphi) \) are totally \( \varphi \)-invariant.

**Corollary 3.4.4** Let \( L \) be a complete orthomodular lattice and \( \varphi \) be an automorphism of \( L \) such that \( \tilde{C}(L \varphi) \) is atomic. If all atoms of \( \tilde{C}(L \varphi) \) are atoms of \( L \) then \( \varphi \) is spectral.

### 3.5 C-maximal Boolean subalgebras of an OML

**Definition 3.5.1** Let \( L \) be an orthomodular lattice. A Boolean subalgebra \( B \subseteq L \) satisfying \( \tilde{C}(K(B)) \subseteq B \) is said to be C-maximal (i.e. maximal with respect to its commutant).

**Theorem 3.5.2** A Boolean subalgebra of an orthomodular lattice is C-maximal if and only if it coincides with its bicommutant.

**Theorem 3.5.3** If the automorphism \( \varphi \) is spectral, then the following assertions are true:

1. \( \tilde{C}(L \varphi) = K(L \varphi) \);
2. \( \tilde{C}(L \varphi) \) is C-maximal.

### 3.6 Spectral automorphisms and physical theories

Let us consider a theory represented by an orthomodular lattice \( L \), which is atomic, complete and has the covering property. Assume also that this theory has a spectral automorphism \( \varphi \) whose spectrum is atomic. Examples of such theories exist, such as a finite dimensional quantum logic. Then, we can construct a basis of \( L \), whose elements are invariant under \( \varphi \). This result is an analogue of the spectral theorem for unitary operators having purely point spectrum.

### 3.7 Spectral automorphisms and Piron’s theorem

Piron’s representation theorem (see [51]) allows us to consider that non-classical theories are based on the Hilbert space formalism. However, a question remains: is the Hilbert space real, complex or quaternionic?

We show that, if there are physical motivations for admitting that spectral symmetries (other than simple reflexions relative to a hyperplane) must exist in a theory, then the real Hilbert spaces have to be excluded from those able to support quantum theories.
3.8 Spectral families of automorphisms and a Stone-type theorem

Definition 3.8.1 Let $L$ be an orthomodular lattice and $\Phi$ be a family of automorphisms of $L$. The family $\Phi$ is spectral if there is a Boolean subalgebra $B$ of $L$ such that:

\[(\varphi(a) = a \text{ for every } \varphi \in \Phi) \text{ if and only if } a \leftrightarrow B.\] (P2)

A Boolean algebra $B$ satisfying condition (P2) is a spectral algebra of $\Phi$. The set of $\Phi$-invariant elements of $L$ (which is a subalgebra of $L$) is denoted by $L_\Phi$.

Proposition 3.8.2 For every spectral family $\Phi$ of automorphisms of an orthomodular lattice $L$ there exists the greatest spectral algebra of the family $\Phi$.

Definition 3.8.3 Let $\Phi$ be a spectral family of automorphisms of an orthomodular lattice. The spectrum $\sigma_\Phi$ of the family $\Phi$ is the greatest spectral algebra of the family $\Phi$.

Proposition 3.8.4 Let $\Phi$ be a spectral family of automorphisms of an orthomodular lattice $L$. Then:

1. $\sigma_\Phi = \bar{C}(L_\Phi)$;
2. $\sigma_\Phi = \bar{C}(K(\sigma_\Phi))$ (i.e., $\sigma_\Phi$ is $C$-maximal);
3. $\sigma_\Phi = K(K(\sigma_\Phi))$.

Theorem 3.8.5 Let $L$ be an orthomodular lattice and $\Phi$ be a family of spectral automorphisms of $L$. Then $\Phi$ is a spectral family if and only if $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, the spectrum $\sigma_\Phi$ of the family contains all spectra $\sigma_\varphi$, $\varphi \in \Phi$.

The purpose of introducing and studying spectral automorphisms has been to construct something similar to the Hilbert space spectral theory without using the specific instruments available in a Hilbert space setting, but using only the abstract orthomodular lattice structure. The next result is intended as an analogue of the Stone theorem concerning strongly continuous uniparametric groups of unitary operators.

Theorem 3.8.6 Let $L$ be an orthomodular lattice and $\Phi$ be a family of spectral automorphisms of $L$. If $\Phi$ is an Abelian group and $\varphi(L_\psi) = L_{\varphi_\psi}$ for every $\varphi, \psi \in \Phi$ with $\psi \notin \{\text{id}, \varphi^{-1}\}$, then:

1. $L_\varphi = L_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\text{id}\}$;
2. $\sigma_\varphi = \sigma_\psi$ for every $\varphi, \psi \in \Phi \setminus \{\text{id}\}$;
3. $\Phi$ is a spectral family.

Let us remark that Theorem 3.8.6 gives purely algebraic conditions for a family of automorphisms to have a spectrum. The last hypothesis (namely, that $\varphi(L_\psi) = L_{\varphi_\psi}$) can be seen as a replacement for the continuity condition in the original Stone theorem.

Preliminaries: unsharp quantum logics

In this second part of the article, we shall move our investigations from the framework of orthomodular posets or lattices—which may be considered as representing “sharp” quantum logics—to the more general framework of effect algebras—regarded as “unsharp” quantum logics. Before immersing into the mathematical universe of effect algebras, let us explain
briefly why they have become important in the contemporary theory of quantum measurement. The reader interested in a more detailed physical interpretation of the mathematical notions presented here should refer to e.g. [4, 5, 42, 45, 44]. An interesting description of how a quantum system with all its ingredients like observables, states and symmetries can be represented using effect algebras (with compression bases) can also be found in [22].

According to the operational approach to quantum mechanics (see, e.g., [45, 44, 42]), states of a quantum system, which intuitively correspond to a complete knowledge of the system, are described in terms of preparation procedures (or classes of them). Moreover, effects, which can be thought of as yes-no measurements that may be unsharp, are defined as equivalence classes of so-called effect apparatuses—i.e., instruments that perform yes-no measurements. As we shall see, we can regard the measurement of any observable as a combination of yes-no measurements—which is why effects play such an important role in the modern theory of quantum measurements.

According to the Hilbert space formalism of quantum mechanics, the various quantities and relations pertaining to the quantum system are representable in terms of operators defined on a complex separable Hilbert space $H$—the so-called state space of the system. More precisely:

- states are represented by density operators $\rho$ on $H$, i.e., trace class operators of trace 1.
- effects are represented by effect operators $A$ on $H$, i.e., elements of the set $\mathcal{E}(H)$ of self-adjoint operators on $H$ lying between the null operator and the identity operator. Let us remark that, in conventional quantum mechanics, yes-no measurements were considered to be represented by projection operators on $H$, the so-called “decision effects” [45], which form a proper subset $\mathcal{P}(H)$ of $\mathcal{E}(H)$ (see also Example 4.2.6). While projections are interpreted as “sharp” events, the effects can be “unsharp” or “fuzzy”, and for instance may fail to satisfy the principle of excluded middle, since the greatest lower bound of effects $A$ and $A'$ (“non-$A$”) may be different from the null effect. Motivation for replacing projection operators with the more general effect operators in representing quantum effects is by now well established. We refer to, e.g., [4, 42]. An argument concerning this matter is also presented in [8, Chapter 5], in connection to the notion of sequential product.
- the probability for the occurrence of an effect $A$ (denoted with the same letter as the corresponding operator) when the system is prepared in a state $\rho$ (again, denoted with the same letter as the corresponding density operator) is $p_{\rho}(A) = \text{tr}(\rho A)$ (where $\text{tr}()$ denotes, of course, the trace).
- observables are represented as normalised positive operator valued measures (POV-measures). A POV-measure is a mapping $E : \mathcal{F} \to \mathcal{E}(H)$, where $(X, \mathcal{F})$ is a measurable space (with $X$ interpreted as the set of possible outcomes of the measurements performed on the observable), mapping which satisfies $E(X) = \mathbf{1}$ and $E(\bigcup M_i) = \sum E(M_i)$ for all disjoint sequences $(M_i)$ in $\mathcal{F}$ (the series converges in weak operator topology). For comparison, let us remark that, according to conventional quantum mechanics, observables were defined as projection valued measures (PV-measures), which can be regarded as a particular case of POV-measures taking values in the set $\mathcal{P}(H)$. It should be mentioned that usually, the measurable space $(X, \mathcal{F})$ is just the real Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, in which case a PV-measure is uniquely associated, by the spectral theorem, to a self-adjoint operator. This allows, for “conventional” observables defined as PV-measures to be represented by (or identified with) self-adjoint operators on the state space $H$. One of the interesting consequences of the transition from conventional observables, defined as self-adjoint operators or the corresponding PV-measures to the
observables defined as POV-measures is the possibility of measuring together observables that do not commute (see [43]).

The interpretation of the above representation of observables as POV-measures is as follows (see [42, 22, 4]). \(X\) represents the set of possible outcomes of the measurements performed on the observable. For a set \(M \in \mathcal{F}\), the question if a measurement of the observable yields a value contained in \(M\) or not is a yes-no measurement, thereby described by an effect operator \(E(M)\). If the result of the measurement performed on the observable belongs to \(M\), we say that the effect \(E(M)\) is observed, whereas in the opposite case it is non-observed. In both cases, \(E(M)\) is tested by the observable, since it is contained in its range. The probability of the measured value of the observable to be in \(M\) when the system is in the state \(\rho\) is \(p_{\rho}(E(M)) = \text{tr}(\rho E(M))\). Therefore, the map \(E : M \mapsto E(M)\) completely describe the “statistics” of the observable in any state \(\rho\) of the system.

A set of effects is simultaneously testable or coexistent if there exist an observable whose range includes it [45, 43]. Later on, we shall discuss in details the properties of the important notion of coexistence in abstract effect algebras, which is the generalization of compatibility/commutativity as defined in orthomodular posets and orthomodular lattices.

The set \(\mathcal{E}(H)\) of effects can be endowed with a family of morphisms \((J_P)_{P \in \mathcal{P}(H)}\) indexed by the projection operators \(P \in \mathcal{P}(H)\) and defined by \(J_P(A) = PAP\), called compressions. They interact in useful ways with states, observables and symmetries. The family \((J_P)_{P \in \mathcal{P}(H)}\) is said to form a compression base of \(\mathcal{E}(H)\).

Just like the structure of orthomodular lattice was introduced as an abstraction of the important features of the lattice \(\mathcal{P}(H)\) of projection operators of a Hilbert space \(H\), the algebraic structure of effect algebra is an abstraction of the essential features of the set \(\mathcal{E}(H)\) of effect operators and compression base effect algebras (CB-effect algebras) are an abstraction of the essential features of the set \(\mathcal{E}(H)\) endowed with the family of morphisms \((J_P)_{P \in \mathcal{P}(H)}\). Other structures that generalize \(\mathcal{E}(H)\) are D-posets, introduced by F. Köpka and F. Chovanec [41] and weak orthoalgebras, introduced by R. Giuntini and H. Greuling [25]. They all turned out to be equivalent structures. In order to avoid confusion and make our exposition easier to follow, we will present all the results in terms of the effect algebra structure, even those that were originally formulated by their authors in terms of another equivalent structure.

We conclude these introductory remarks with the words of D. Foulis [22], which, we believe, could best motivate our study of CB-effect algebras:

“A physical system \(S\), understood as the subject of experimental investigation, is appropriately represented by a CB-effect algebra \(E\), which hosts the observables, states, symmetries and other ingredients of a physical theory for \(S\). Moreover, physical systems are classified by the CB-effect algebras that represent them.”

4 Basics on effect algebras

This section is devoted to an introduction to effect algebras and their basic properties. The facts presented here can be found, e.g., in [23, 24, 25, 27, 41, 15].

4.1 Effect algebras. Basic definitions and properties

Definition 4.1.1 An effect algebra is an algebraic structure \((E, \oplus, 0, 1)\) such that \(E\) is a set, \(0\) and \(1\) are distinct elements of \(E\) and \(\oplus\) is a partial binary operation on \(E\), and the
following conditions hold for every \(a, b, c \in E\) (the equalities should be understood in the sense that if one side exists, the other side exists as well):

**(EA1)** \(a \oplus b = b \oplus a\) (commutativity)

**(EA2)** \((a \oplus b) \oplus c = a \oplus (b \oplus c)\) (associativity)

**(EA3)** for every \(a \in E\), there exists a unique \(a' \in E\) such that \(a \oplus a' = 1\) (orthosupplement)

**(EA4)** if \(a \oplus 1\) is defined, then \(a = 0\) (zero-unit law).

An orthogonality and a partial order relation are defined in an effect algebra as follows:

**Definition 4.1.2** Let \(E\) be an effect algebra. Elements \(a, b \in E\) are called *orthogonal* (denoted by \(a \perp b\)) if the sum \(a \oplus b\) is defined. We write \(a \leq b\) if there is an element \(c \in E\) such that \(a \oplus c = b\).

The next couple of propositions gives a list of basic properties that hold in effect algebras.

**Proposition 4.1.3** Let \(E\) be an effect algebra. For every \(a, b \in E\) the following properties hold:

1. \(a'' = a\)
2. \(a \leq b\) implies \(b' \leq a'\)
3. \(1' = 0\) and \(0' = 1\).

**Proposition 4.1.4** Let \(E\) be an effect algebra. For every \(a, b, c \in E\) the following properties hold:

1. \(0 \leq a \leq 1\).
2. \(a \oplus 0 = a\).
3. \(a \perp b\) if and only if \(a \leq b'\).
4. If \(a \leq b\) and \(c \in E\) is such that \(a \oplus c = b\), then \(c\) is uniquely determined by the elements \(a\) and \(b\), namely \(c = (a \oplus b')'\). We will then denote \(c = b \ominus a\).
5. “\(\leq\)” is a partial order on \(E\).
6. \(a \oplus b = a \oplus c\) implies \(b = c\) (cancellation law).
7. \(a \oplus b \leq a \oplus c\) implies \(b \leq c\) (cancellation law).

**Definition 4.1.5** Let \(E\) be an effect algebra and \(F \subset E\). If \(0, 1 \in F\), and \(F\) is closed to \(\oplus\) and to orthosupplementation, then \((F, \oplus|_{F \times F}, 0, 1)\) is a sub-effect algebra of \(E\).

### 4.2 Special elements. Coexistence

**Definition 4.2.1** An element \(a\) of an effect algebra \(E\) is called:

- *isotropic* if \(a \perp a\);
- *sharp* \((a \in E_S)\) if \(a \wedge a' = 0\);
- *principal* if for every orthogonal pair \(b, c \in E\), \(b, c \leq a\) we have \(b \oplus c \leq a\);
- *central* if \(a, a'\) are principal and for every \(b \in E\), there are \(b_1, b_2 \in E\) such that \(b_1 \leq a, b_2 \leq a'\) and \(b = b_1 \oplus b_2\).

**Proposition 4.2.2** In an effect algebra, the following assertions hold:

1. every central element is principal;
2. every principal element is sharp;
3. every nonzero sharp element is nonisotropic.
In general, the converse statements do not hold.

**Definition 4.2.3** An orthoalgebra is an effect algebra whose only isotropic element is $0$.

**Definition 4.2.4** Let $E$ be an effect algebra and let us denote by $na$ the sum of $n$ copies of an element $a \in E$, if it exists. We call $E$ Archimedean if $\sup\{n \in \mathbb{N} : na \text{ is defined}\} < \infty$ for every nonzero element $a \in E$.

**Remark 4.2.5** Let $(E, \leq', 0, 1)$ be an orthomodular poset and define $a \oplus b = a \lor b$ for every orthogonal (i.e. $a \leq b'$) pair of elements $a, b \in E$. It is a routine verification that $(E, \oplus, 0, 1)$ is an effect algebra (an orthoalgebra even) and, moreover, the order and supplement in the effect algebra coincide with the order and complement in the orthomodular poset.

**Example 4.2.6** Let us now present the prototypical example of effect algebra. Let $H$ be a separable, complex Hilbert space and let $0, 1$ denote the zero and identity operators on $H$. An ordering is defined on the set of bounded self-adjoint operators on $H$ by:

$$A \leq B \iff \langle Ax, x \rangle \leq \langle Bx, x \rangle \text{ for all } x \in H$$

(4.1)

where $\langle \cdot, \cdot \rangle$ denotes of course the inner product on $H$. Then $\mathcal{E}(H)$ is defined as the set of self-adjoint operators on $H$ lying between $0$ and $1$, in the sense of (4.1) – the so-called effect operators. A partial binary operation “$\oplus$” is defined on $\mathcal{E}(H)$ by $A \oplus B = A + B$, whenever $A + B \in \mathcal{E}(H)$. Then $(\mathcal{E}(H), \oplus, 0, 1)$ is the so-called standard Hilbert space effect algebra. The supplement of an effect operator $A \in \mathcal{E}(H)$ is $A' = 1 - A$. Let us remark that the effect algebra order relation defined on $\mathcal{E}(H)$ according to Definition 4.1.2 coincides with the order relation defined in (4.1). It is worth noting that there are nonzero isotropic effects, e.g. $\frac{1}{2}1 \in \mathcal{E}(H)$ (which is even its own supplement). Therefore, $\mathcal{E}(H)$ is not an orthoalgebra. Its sharp elements are in fact the projection operators (self-adjoint idempotents) on $H$, which form an orthomodular lattice which is also a sub-effect algebra of $\mathcal{E}(H)$, denoted, as previously mentioned, by $\mathcal{P}(H)$.

**Definition 4.2.7** Let $E$ be an effect algebra and $a, b \in E$. Elements $a$ and $b$ coexist in $E$ if there are $a_1, b_1, c \in E$ such that $a = a_1 \lor c$, $b = b_1 \lor c$ and $a_1 \lor b_1 \lor c$ exists in $E$. In this case, we write $a \leftrightarrow b$. For a subset $M$ of $E$, we write $a \leftrightarrow M$ if $a \leftrightarrow b$ for all $b \in M$. The commutant of $M$ in $E$ is the set $K_E(M) = \{a \in E : a \leftrightarrow M\}$. If there is no possibility of confusion concerning $E$, we shall simply denote it by $K(M)$.

**Proposition 4.2.8** Let $E$ be an effect algebra. Then:

1. an element $a \in E$ is central if and only if it coexists with all elements of $E$ and $a, a'$ are principal;
2. if $E$ is an orthomodular poset, the set of central elements of $E$ coincides with the center of $E$ as an orthomodular poset.

**Remark 4.2.9** Coexistence generalizes compatibility to effect algebras (see Theorem 4.2.11). The two notions coincide in orthomodular posets. This justifies the use of the same notation for coexistence and compatibility and also for the commutant with respect to coexistence or compatibility.

The notion of center of an effect algebra, defined as the set of its central elements, generalizes the notion of center in orthomodular posets, defined as the set of its elements.
which are compatible with all the others. We shall denote the center of an effect algebra $E$ by $\tilde{C}(E)$.

**Theorem 4.2.10** [27, Theorem 5.4] The center $\tilde{C}(E)$ of an effect algebra $E$ is a sub-effect algebra of $E$ and as an effect algebra in its own right, $\tilde{C}(E)$ forms a Boolean algebra. Furthermore, if $a, b \in \tilde{C}(E)$, then $a \land b$ and $a \lor b$ as calculated in $\tilde{C}(E)$ are also the infimum and supremum of $a$ and $b$ as calculated in $E$.

**Theorem 4.2.11** [see [23, 24]] An effect algebra is:

1. an orthoalgebra if and only if its every element is sharp if and only if $a \oplus b$ is a minimal upper bound of $a, b$ for every orthogonal pair $a, b \in E$;
2. an orthomodular poset if and only if its every element is principal if and only if $a \oplus b = a \lor b$ for every orthogonal pair $a, b \in E$;
3. a Boolean algebra if and only if its every element is central.

### 4.3 Substructures in effect algebras

**Definition 4.3.1** A Boolean subalgebra of an effect algebra $E$ is a sub-effect algebra of $E$ which is a Boolean algebra with $'$ and with the operations $\lor, \land$ induced by the order in $E$.

**Proposition 4.3.2** Let $E$ be an orthoalgebra and let $F \subseteq E$ be a Boolean subalgebra of $E$.

Then:

1. If $a, b \in F$ and $a \land_E b$ exists, then $a \land_E b = a \land_F b$;
2. If $a, b \in F$ and $a \lor_E b$ exists, then $a \lor_E b = a \lor_F b$;

### 4.4 Important classes of effect algebras

We present a few important properties that an effect algebra may fulfill and their correlations.

**Definition 4.4.1** An effect algebra that is a lattice with respect to its usual order relation, is called a lattice effect algebra.

**Definition 4.4.2** Let $E$ be an effect algebra. A system $(a_i)_{i \in I}$ of elements of $E$ is orthogonal if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subseteq I$. A majorant of an orthogonal system $(a_i)_{i \in I}$ is an upper bound of $\{\bigoplus_{i \in F} a_i : F \subseteq I$ is finite\}. The sum of an orthogonal system is its least majorant (if it exists).

**Definition 4.4.3** An effect algebra $E$ is orthocomplete if every orthogonal system of its elements has a sum. An effect algebra $E$ is weakly orthocomplete if every orthogonal system in $E$ has a sum or no minimal majorant.

**Definition 4.4.4** An effect algebra $E$ has the maximality property if the set $\{a, b\}$ has a maximal lower bound for every $a, b \in E$.

**Remark 4.4.5** The maximality property was introduced by Tkadlec [59]. In [62, Theorem 2.2], he proved that effect algebras with the maximality property or the ones that are weakly orthocomplete are common generalizations of lattice effect algebras and orthocomplete effect algebras. Since finite or chain-finite effect algebras are orthocomplete (see [61, Theorem 4.1]), they also must satisfy the maximality property.
Definition 4.4.6  $E$ is determined by atoms if, for different $a, b \in E$, the sets of atoms dominated by $a$ and $b$ are different.

Lemma 4.4.7  Every atomistic effect algebra is determined by atoms. Every effect algebra determined by atoms is atomic.

Examples showing that the converse implications do not hold can be found in [26, 50].

4.5 Morphisms of effect algebras

Definition 4.5.1  Let $E$ and $E'$ be effect algebras and let $\varphi : E \to E'$ be a map. We call $\varphi$ an additive map if $a \perp b$ implies $\varphi(a) \perp \varphi(b)$ and $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$, for every $a, b \in E$.

We call $\varphi$ a morphism of effect algebras if it is additive and $\varphi(1_E) = 1_{E'}$. A morphism $\varphi$ of effect algebras which preserves the infimum (i.e., $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$, whenever $a \wedge b$ exists) is a $\wedge$-morphism. A bijective morphism $\varphi$ such that $\varphi^{-1}$ is also a morphism is an isomorphism. An isomorphism $\varphi : E \to E$ is an automorphism.

Proposition 4.5.2  Let $E$ and $E'$ be effect algebras and let $\varphi : E \to E'$ be a map. Then $\varphi$ is an isomorphism of effect algebras if and only if it is bijective and, for every $a, b \in E$, $a \perp b$ if and only if $\varphi(a) \perp \varphi(b)$, in which case $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$. Moreover, if $\varphi$ is an isomorphism, it is also a $\wedge$-morphism.

5 Sequential, compressible and compression base effect algebras

In this section, we present the important established facts concerning sequential, compressible and compression base effect algebras, laying the foundation for the results that will be the presented in the following sections. Our presentation is based on [29, 30, 32, 55].

5.1 Sequential effect algebras

The notion of a sequential product defined in general effect algebras was introduced by Gudder and Greechie [32]. This sequential product satisfies a set of physically motivated axioms as it formalizes the case of sequentially performed measurements.

Definition 5.1.1 A sequential product on an effect algebra $(E, \oplus, 0, 1)$ is a binary operation $\circ$ on $E$ such that for every $a, b, c \in E$, the following conditions hold:

(S1) $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$ if $b \oplus c$ exists;
(S2) $1 \circ a = a$;
(S3) if $a \circ b = 0$ then $a | b$ (where $a | b$ denotes $a \circ b = b \circ a$);
(S4) if $a | b$ then $a | b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$;
(S5) if $c | a, b$ then $c | a \circ b$ and $c | (a \oplus b)$ (if $a \oplus b$ exists).

An effect algebra $E$ endowed with a sequential product is called a sequential effect algebra.

5.2 Compressible effect algebras

Definition 5.2.1  Let $E$ be an effect algebra and $J : E \to E$ be an additive map. If $a \leq J(1)$ implies $J(a) = a$, then $J$ is a retraction. In this case, $J(1)$ is the focus of $J$. If, moreover,
\( J(a) = 0 \) implies \( a \leq J(1)' \), then \( J \) is a compression. The focus of a retraction on \( E \) is a projection.

**Proposition 5.2.2** [see [29, Lemma 3.1, Lemma 3.2, Lemma 3.3]] Let \( E \) be an effect algebra and \( J : E \to E \) be a compression with focus \( p \). Then:

1. \( J \) is idempotent;
2. \( J \) preserves order;
3. \( \text{Ker}(J) = [0, p'] \);
4. \( J(E) = [0, p] \);
5. \( p \) is principal and therefore sharp;

**Definition 5.2.3** An effect algebra \( E \) is compressible if every retraction on \( E \) is a compression and it is uniquely determined by its focus.

**Remark 5.2.4** If \( E \) is a sequential effect algebra, the sequential product with a sharp (and therefore principal) element \( p \in E_S \) defines a compression with focus \( p \) by \( J_p(a) = p \circ a \) [30]. If, moreover, \( E \) is compressible, then \( J_p : E \to E \), \( J_p(a) = p \circ a \) is the unique compression on \( E \) with focus \( p \). The close relation between sequential and compressible effect algebras becomes now evident.

### 5.3 Compression bases in effect algebras

Effect algebras with compression bases are a common generalization of compressible and sequential effect algebras. Our presentation of compression base effect algebras is based on [30, 55].

**Definition 5.3.1** Let \( E \) be an effect algebra. A sub-effect algebra \( F \) of \( E \) is normal if, for every \( a, b, c \in E \) such that \( a \oplus b \oplus c \) exists in \( E \) and \( a \oplus b, b \oplus c \in F \), it follows that \( b \in F \).

**Definition 5.3.2** Let \( E \) be an effect algebra. A system \( (J_p)_{p \in P} \) of compressions on \( E \) indexed by a normal sub-effect algebra \( P \) of \( E \) is called a compression base for \( E \) if the following conditions hold:

1. Each compression \( J_p \) has the focus \( p \).
2. If \( p, q, r \in P \) and \( p \oplus q \oplus r \) is defined in \( E \), then \( J_{p \oplus r} \circ J_{r \oplus q} = J_r \).

**Theorem 5.3.3** [30, Theorems 3.3 and 3.4]

1. If \( E \) is a compressible effect algebra, then the set \( P(E) \) of its projections is a normal sub-effect algebra of \( E \) and \( (J_p)_{p \in P(E)} \) is a compression base for \( E \).
2. If \( E \) is a sequential effect algebra, then the set \( E_S \) of its sharp elements is a normal sub-effect algebra of \( E \). If, for every \( p \in E_S \), \( J_p \) is the compression on \( E \) defined by \( J_p(a) = p \circ a \), for every \( a \in E \), then \( (J_p)_{p \in E_S} \) is a maximal compression base for \( E \).

For an effect algebra \( E \) with a compression base \( (J_p)_{p \in P} \) we will maintain, from now on, the following notations:

- \( p \circ a = J_p(a) \) for every \( p \in P \) and \( a \in E \);
- \( p \mid q \) if \( p, q \in P \) and \( p \circ q = q \circ p \) (i.e., \( J_p(q) = J_q(p) \));
- \( C(p) = \{ a \in E : a = J_p(a) \oplus J_{p'}(a) \} \) for every \( p \in P \).
Definition 5.3.4 A compression base \((J_p)_{p \in P}\) on the effect algebra \(E\) has the projection cover property if for every element \(a \in E\) there exists the least element \(b \in P\) (the projection cover of \(a\)) with \(b \geq a\).

Theorem 5.3.5 [see [55, Theorem 5.1]] Let \(E\) be an effect algebra with a compression base \((J_p)_{p \in P}\) that has the projection cover property. Then \(P\) is an orthomodular lattice.

6 Spectral automorphisms in CB-effect algebras

In the third section we have introduced spectral automorphisms (see also [37]). They resulted from our attempt to construct, in the abstract framework of orthomodular lattices, an analogue of the spectral theory in Hilbert spaces. We generalize spectral automorphisms to the framework of compression base effect algebras, currently considered as the appropriate mathematical structures for representing physical systems [22]. The results presented here have been published in [9].

6.1 Spectral automorphisms: the idea and definitions

Let \(H\) be a Hilbert space and \(\mathcal{E}(H)\) the corresponding standard effect algebra. Automorphisms of \(\mathcal{E}(H)\) are of the form \(\varphi_U : \mathcal{E}(H) \to \mathcal{E}(H), \varphi_U(A) = UAU^{-1}\), where \(U\) is a unitary or antiunitary Hilbert space operator [22]. An element \(A \in \mathcal{E}(H)\) is \(\varphi_U\)-invariant if and only if \(\varphi_U(A) = UAU^{-1} = A\), i.e., operators \(U\) and \(A\) commute. Let \(B_U\) be the Boolean algebra of projection operators that is the image of the projection-valued spectral measure associated to \(U\). Then, operators \(A\) and \(U\) commutes if and only if \(A\) commutes with \(B_U\) (i.e., with every projection operator in \(B_U\)) [35]. We are therefore led to the following definition of spectral automorphisms in compression base effect algebras:

Definition 6.1.1 Let \(E\) be an effect algebra and \((J_p)_{p \in P}\) be a compression base for \(E\). An automorphism \(\varphi : E \to E\) is spectral if there exists a Boolean subalgebra \(B\) of \(P\) with the property:

\[
\varphi(a) = a \text{ if and only if } a \leftrightarrow B
\]

(P1)

Proposition 6.1.2 Let \(E\) be an effect algebra, \((J_p)_{p \in P}\) be a compression base for \(E\) and \(\varphi : E \to E\) be a spectral automorphism. There exists the greatest Boolean subalgebra \(B \subseteq P\) satisfying (P1).

Definition 6.1.3 Let \(E\) be an effect algebra, \((J_p)_{p \in P}\) be a compression base for \(E\) and \(\varphi : E \to E\) be a spectral automorphism. The greatest Boolean subalgebra of \(P\) fulfilling (P1) is the spectrum of the automorphism \(\varphi\), denoted by \(\sigma^P_{\varphi}\).

Proposition 6.1.4 Let \(E\) be an effect algebra and \((J_p)_{p \in P}\) be a compression base for \(E\). If \(P \subseteq \hat{C}(E)\), then the identity is the only spectral automorphism of \(E\).

Remark 6.1.5 As a particular case, if \(E\) is a Boolean algebra, then its identity is its only spectral automorphism. Therefore, the presence of nontrivial spectral automorphisms allows us to distinguish between classical (Boolean) and nonclassical theories.
6.2 Characterizations and properties of spectral automorphisms

For an automorphism \( \varphi \) of an effect algebra \( E \), we will denote by \( E_\varphi \) the set of \( \varphi \)-invariant elements of \( E \). Due to the definition properties of automorphisms, it is clear that \( E_\varphi \) is a sub-effect algebra of \( E \).

The following lemma and corollary, that will be useful in the sequel, are related to \cite{27} Theorem 4.2 and Lemma 5.2. However, the statements we prove are slightly more general and could be interesting in their own right.

**Lemma 6.2.1** Let \( E \) be an effect algebra, \( \{e_1, e_2, \ldots, e_n\} \) be an orthogonal set of its elements (i.e., the sum \( \bigoplus_{i=1}^{n} e_i \) exists) and consider \( p \in E \) such that \( p = \bigoplus_{i=1}^{n} p_i \) with \( p_i \leq e_i \). If \( e_j \) is principal for some \( j \in \{1, 2, \ldots, n\} \), then \( p \land e_j \) exists in \( E \) and \( p_j = p \land e_j \).

**Corollary 6.2.2** If \( a, a' \) are principal elements of the effect algebra \( E \), \( b \in E \) and \( a \leftrightarrow b \), then \( a \land b \) and \( a' \land b \) exist in \( E \) and \( b = (a \land b) \oplus (a' \land b) \).

**Theorem 6.2.3** Let \( E \) be an effect algebra and \( (J_p)_{p \in P} \) be a compression base for \( E \). If \( \varphi : E \to E \) is a spectral automorphism, then \( \sigma_\varphi^P = \tilde{C}(E_\varphi) \cap P \).

**Corollary 6.2.4** Let \( E \) be an effect algebra, \( (J_p)_{p \in P} \) be a compression base for \( E \) and \( \varphi : E \to E \) be an automorphism. Then \( \varphi \) is spectral if and only if \( K(\tilde{C}(E_\varphi) \cap P) \subseteq E_\varphi \) (the converse inclusion is always true).

**Theorem 6.2.5** Let \( E \) be an effect algebra, \( (J_p)_{p \in P} \) be a compression base for \( E \) and \( \varphi : E \to E \) be an automorphism. Then \( \varphi \) is spectral if and only if \( a \land b \in E_\varphi \) for every \( a \in \tilde{C}(E_\varphi) \cap P, b \in K(\tilde{C}(E_\varphi) \cap P) \).

The search for the conditions that a Boolean algebra must fulfill in order to be the spectrum of a spectral automorphism leads to the following notion.

**Definition 6.2.6** Let \( E \) be an effect algebra and \( (J_p)_{p \in P} \) be a compression base for \( E \). A Boolean subalgebra \( B \subseteq P \) is \( C \)-maximal if \( \tilde{C}(K(B)) \cap P \subseteq B \).

**Theorem 6.2.7** Let \( E \) be an effect algebra and \( (J_p)_{p \in P} \) be a compression base for \( E \). A Boolean subalgebra \( B \subseteq P \) is \( C \)-maximal if and only if \( B = K(K(B)) \cap P \).

**Corollary 6.2.8** Let \( E \) be an effect algebra, \( (J_p)_{p \in P} \) be a compression base for \( E \) and \( \varphi : E \to E \) be a spectral automorphism. Then:

1. \( \sigma_\varphi^P = \tilde{C}(E_\varphi) \cap P \) is \( C \)-maximal;
2. \( \sigma_\varphi^P = K(K(\sigma_\varphi^P)) \cap P \);
3. \( \sigma_\varphi^P = K(E_\varphi) \cap P \).

6.3 An application of spectral automorphisms to \( \mathcal{E}(H) \)

The notion of spectral automorphism was introduced with the declared intention to obtain an analogue of the Hilbert space spectral theory in the abstract setting of compression base effect algebras. It is time to see if this attempt was successful, by applying the abstract theory to the particular case of the standard Hilbert space effect algebra. Therefore, we devote this
section to the proof of a “spectral theorem” in $\mathcal{E}(H)$, the set of self-adjoint operators between the null and the identity operators, for a finite-dimensional Hilbert space $H$.

Let us denote in the sequel by $\hat{e}$ the 1-dimensional subspace generated by $e \in H$, $\|e\| = 1$ and $P_e$ the corresponding projection operator, i.e., $P_e : H \to H$, $P_e x = \langle x, e \rangle e$ (where $\langle \cdot, \cdot \rangle$ denotes the inner product of $H$).

**Theorem 6.3.1** Let $H$ be an n-dimensional Hilbert space, $\mathcal{E}(H)$ be its standard effect algebra and $(J_P)_{P \in \mathcal{P}(H)}$ be the canonical compression base for $\mathcal{E}(H)$. Let $U : H \to H$ be a unitary operator and $\varphi : \mathcal{E}(H) \to \mathcal{E}(H)$ be the automorphism defined by $\varphi(A) = UAU^{-1}$. If $\varphi$ is spectral, then:

1. There is an orthogonal basis $\{e_1, e_2, \ldots, e_n\}$ of $H$ such that for every $i \in \{1, 2, \ldots, n\}$, $U e_i = \lambda_i e_i$ where $\lambda_i$ is a scalar, $|\lambda_i| = 1$.
2. There exists a partition $\Pi$ of the set $\{1, 2, \ldots, n\}$ such that any $\varphi$-invariant atom of $\mathcal{P}(H)$ is a 1-dimensional subspace in exactly one of the subspaces $\bigvee_{j \in J} \hat{e}_j$, $J \in \Pi$.
3. If the subalgebra $\mathcal{E}(H)_{\varphi}$ of $\varphi$-invariant elements of $\mathcal{E}(H)$ is Boolean, then the spectrum $\sigma^P_{\varphi}\mathcal{E}(H) = \mathcal{E}(H)_{\varphi} \cap \mathcal{P}(H)$ is a block in $\mathcal{P}(H)$. In this case all eigenvalues of $U$ are distinct and $\Pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$.
4. The spectrum $\sigma^P_{\varphi}\mathcal{E}(H)$ is the Boolean algebra generated by $\{\bigvee_{j \in J} \hat{e}_j : J \in \Pi\}$.
5. If the effect $A \in \mathcal{E}(H)$ is $\varphi$-invariant and $P \in \mathcal{P}(H)$ is the smallest projection that dominates $A$ (namely the projection on the range of $A$), then $P$ is $\varphi$-invariant too.
6. If $A$ is a $\varphi$-invariant nonzero effect dominated by an atom of $\mathcal{P}(H)$, then the range of $A$ is included in one of the subspaces $\bigvee_{j \in J} \hat{e}_j$, $J \in \Pi$.

**Remark 6.3.2** The properties (1)–(6) from Theorem 6.3.1 were derived only from the fact that $\varphi$ is spectral, without any other information except for the properties of unitary operators.

### 6.4 Spectral families of automorphisms

Let $E$ denote, for the rest of this section, an effect algebra endowed with a compression base $(J_p)_{p \in P}$ and let $\Phi$ be a family of automorphisms of $E$.

**Definition 6.4.1** The family $\Phi$ of automorphisms of $E$ is called a spectral family of automorphisms if there exists a Boolean subalgebra $B_{\Phi}$ of $P$ satisfying:

$$\varphi(a) = a, \text{ for all } \varphi \in \Phi \text{ if and only if } a \leftrightarrow B_{\Phi} \quad (P2)$$

In the sequel, we denote $E_{\Phi} = \{a \in E : \varphi(a) = a, \text{ for all } \varphi \in \Phi\}$. Let us remark that $E_{\Phi} = \bigcap_{\varphi \in \Phi} E_{\varphi}$ and therefore it’s a sub-effect algebra of $E$.

**Proposition 6.4.2** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a spectral family of automorphisms of $E$. There exists the greatest Boolean subalgebra $B_{\Phi}$ of $P$ satisfying (P2).

**Definition 6.4.3** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a spectral family of automorphisms of $E$. The spectrum (denoted by $\sigma^P_{\Phi}$) of the spectral family $\Phi$ of automorphisms is the greatest Boolean subalgebra $B$ of $P$ fulfilling (P2).

**Theorem 6.4.4** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a spectral family of automorphisms of $E$. Then $\sigma^P_{\Phi} = \tilde{C}(E_{\Phi}) \cap P$.

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**Corollary 6.4.5** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a family of automorphisms of $E$. Then $\Phi$ is a spectral family if and only if $K(\tilde{C}(E_\Phi) \cap P) \subseteq E_\Phi$ (the converse inclusion is trivially satisfied).

**Proposition 6.4.6** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a family of automorphisms of $E$. Then:

1. $\sigma_P^\Phi = \tilde{C}(E_\Phi) \cap P$ is C-maximal;
2. $\sigma_P^\Phi = K(K(\sigma_P^\psi)) \cap P$;
3. $\sigma_P^\Phi = K(E_\Phi) \cap P$.

**Theorem 6.4.7** Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a family of spectral automorphisms of $E$. Then $\Phi$ is a spectral family of automorphisms if and only if the spectra of the automorphisms in the family are pairwise compatible, i.e., $\sigma_\varphi \leftrightarrow \sigma_\psi$ for every $\varphi, \psi \in \Phi$. In this case, $\sigma_P^\Phi$ includes all spectra of automorphisms in the family.

**Theorem 6.4.8** (A “replica” of Stone’s Theorem on strongly continuous uniparametric groups of unitary operators.) Let $E$ be an effect algebra, $(J_p)_{p \in P}$ be a compression base for $E$ and $\Phi$ be a family of spectral automorphisms of $E$. If the following conditions are fulfilled:

1. $\Phi$ is an abelian group;
2. $\varphi(E_\psi) = E_{\varphi\psi}$ for every $\varphi, \psi \in \Phi$ such that $\psi \notin \{id_E, \varphi^{-1}\}$, then:

1. $E_\varphi = E_\psi$ for all $\varphi, \psi \in \Phi \setminus \{id_E\}$;
2. $\sigma_P^\varphi = \sigma_P^\psi$ for all $\varphi, \psi \in \Phi \setminus \{id_E\}$;
3. $\Phi$ is a spectral family.

Let us remark that Theorem 6.4.8 generalizes Theorem 3.8.6 to spectral automorphisms in CB-effect algebras.

7 Atomic effect algebras with compression bases

In the first subsection we establish some properties of atoms in effect algebras endowed with a compression base, mainly regarding coexistence and centrality. Then, in the second subsection, we introduce the notion of projection-atomicity—which is the property of a compression base effect algebra of having the atoms compressions foci. Consequences of projection-atomicity are studied, some of which generalize results obtained in [60]. A few conditions for an atomic compression base effect algebra to be a Boolean algebra are established. Finally, we apply these results to the particular case of sequential effect algebras and find a sufficient condition for them to be Boolean algebras that strengthens previous results by Gudder and Greechie [32] and Tkadlec [60]. The results presented here have been published in [11].

7.1 Atoms and centrality

**Proposition 7.1.1** Let $E$ be an effect algebra. If $p$ is an atom in $E$ that is the focus of a compression and $a \in E$ then $p \leq a$ or $p \leq a'$.

**Corollary 7.1.2** Distinct atoms that are foci of compressions in an effect algebra are orthogonal.
Theorem 7.1.3 Let $E$ be an effect algebra with a compression base $(J_p)_{p \in P}$. If $E$ is determined by atoms and every atom is in $P$ then $P$ is a Boolean algebra.

The conclusion of the above theorem cannot be improved to the statement that $E$ is a Boolean algebra as we show by an example.

Lemma 7.1.4 Let $E$ be an effect algebra with a compression base $(J_p)_{p \in P}$. If $p \in P$ is an atom in $E$ then $C(p) = E$.

Theorem 7.1.5 Let $E$ be an effect algebra with a compression base $(J_p)_{p \in P}$. Every $p \in P$ that is an atom in $E$ is central in $E$.

7.2 Projection-atomic effect algebras

Definition 7.2.1 An effect algebra $E$ is projection-atomic if it is atomic and there is a compression base $(J_p)_{p \in P}$ of $E$ such that $P$ contains all atoms in $E$.

Proposition 7.2.2 Every projection-atomic effect algebra is an orthoalgebra.

Definition 7.2.3 A subset $M$ of an effect algebra $E$ is downward directed if for every $a, b \in M$ there is an element $c \in M$ such that $c \leq a, b$.

An effect algebra $E$ is weakly distributive if $a \wedge b = a \wedge b' = 0$ implies $a = 0$ for every $a, b \in E$.

Theorem 7.2.4 ([59, Theorem 4.2]) Every weakly distributive orthomodular poset with the maximality property is a Boolean algebra.

Lemma 7.2.5 Every projection-atomic effect algebra is weakly distributive.

Lemma 7.2.6 The set of upper bounds of a set of atoms in a projection-atomic effect algebra with the maximality property is downward directed.

Lemma 7.2.7 Every element in a projection-atomic effect algebra is a minimal upper bound of the set of atoms it dominates. Every projection-atomic effect algebra with the maximality property is atomistic.

Lemma 7.2.8 Every projection-atomic effect algebra with the maximality property is an orthomodular poset.

Theorem 7.2.9 Every projection-atomic effect algebra with the maximality property is a Boolean algebra.

We can replace the maximality property in Theorem 7.2.9 by various stronger properties (see Remark 4.4.5), e.g., by the orthocompleteness. It cannot be replaced by the weak orthocompleteness, as we show by an example.

Theorem 7.2.10 Let $E$ be a projection-atomic effect algebra. If a compression base on $E$ for which all atoms are projections has the projection cover property, then $E$ is a Boolean algebra.
Corollary 7.2.11  Every atomic sequential orthoalgebra is a Boolean algebra.

The above corollary generalizes similar results obtained by Gudder and Greechie [32, Theorem 5.3] and Tkadlec [60, Theorems 5.4 and 5.6]. The first mentioned result assumes that the effect algebra is atomistic, the second assumes it has the maximality property and the third assumes it is determined by atoms.

References