

# Minimal and Reduced Reversible Automata

Giovanna J. Lavado, Giovanni Pighizzini, Luca Prigioniero

Dipartimento di Informatica  
Università degli Studi di Milano, Italy

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# Reversible automata

## Motivations and related work

- ▶ Reversibility is a fundamental principle in physics
- ▶ Reversible transformations: (ideal) reversible process in thermodynamics
- ▶ Reversible processes imply no loss of information
- ▶ Logical irreversibility is associated with physical irreversibility and implies a certain amount of heat generation [Landauer '61]

In order to avoid power dissipation and reduce the overall power consumption, the possibility of realizing reversible machines looks appealing!

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## Motivations and related work

Reversibility in different computational devices:

- ▶ Each Turing machine can be simulated by a reversible machine [Bennet '73]
- ▶ Each deterministic machine can be simulated by a reversible machine which uses the same amount of space [Lange&McKenzie&Tapp '00]
- ▶ This implies that each regular language is accepted by a *reversible two-way deterministic finite automaton* [Kondacs '97]

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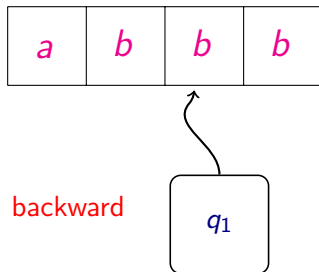
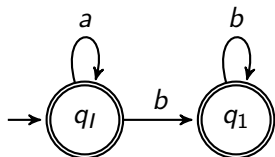
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# Reversible automata

## Motivations and related work

The regular language  $a^*b^*$  cannot be accepted by *any* reversible automaton

[Pin '92]



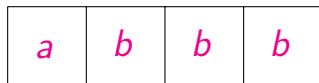
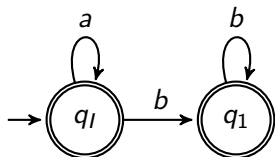
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# Reversible automata

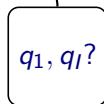
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backward



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# Reversible automata

## Motivations and related work

There are some different notions of reversible automata:

- ▶ *One initial and only one final state* [Angluin '82]
- ▶ *A set of initial states and a set of final states* [Pin '92]
- ▶ *One initial state and a set of final states as in classical automata*

[Holzer&Jakobi&Kutrib '15]

# Reversible automata: preliminaries

Let  $A = (Q, \Sigma, \delta, q_I, F)$  be a *deterministic automaton* (DFA)

We assume:

- ▶ Partial *transition function*  $\delta$
- ▶ Only useful states (no dead state)
- ▶ Unique initial state  $q_I$  and possibly many accepting states

## Reverse transition function

$$\delta^R : Q \times \Sigma \rightarrow 2^Q$$

associates with each state  $r \in Q$  and letter  $a \in \Sigma$  the set of states from which  $r$  can be reached by reading  $a$ , i.e.,

$$\delta^R(r, a) = \{q \in Q \mid \delta(q, a) = r\}$$

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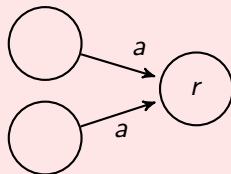
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# Reversible automata: preliminaries

## Irreversible state

A state  $r \in Q$  is said to be *irreversible* when there are at least two transitions on the same letter  $a \in \Sigma$  entering  $r$ , i.e.,

$$\#\delta^R(r, a) \geq 2$$

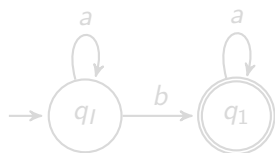


otherwise  $r$  is *reversible*

# Reversible automata

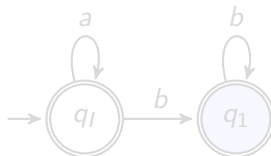
- ▶ A **DFA** is said to be *reversible* (REV-DFA) when *each* state is reversible

reversible automaton



$a^*ba^*$

not reversible automaton

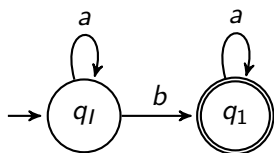


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# Reversible automata

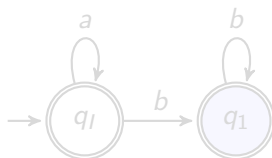
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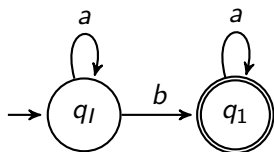


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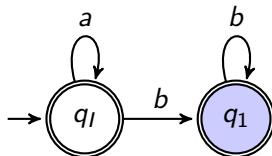
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# Reversible languages

- ▶ A *language* is *reversible* when there *exists* a REV-DFA accepting it

$a^*b^*$  is not a reversible language

How to decide the reversibility of a given regular language?



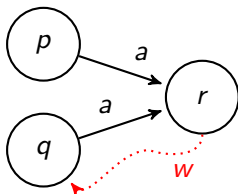
# Reversible languages: a structural characterization

- ▶  $L$  regular language
- ▶  $M = (Q, \Sigma, \delta, q_I, F)$  minimum DFA accepting  $L$

## Theorem ([Holzer&Jakobi&Kutrib '15])

*The language  $L$  is reversible if and only if there do not exist useful states  $p, q \in Q$ , a letter  $a \in \Sigma$ , and a string  $w \in \Sigma^*$  such that  $p \neq q$ ,  $\delta(p, a) = \delta(q, a)$ , and  $\delta(q, aw) = q$ .*

Forbidden pattern:



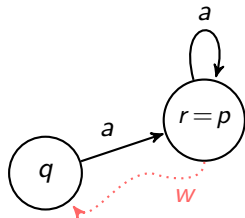
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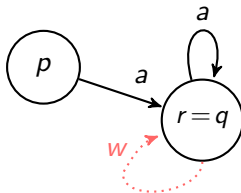
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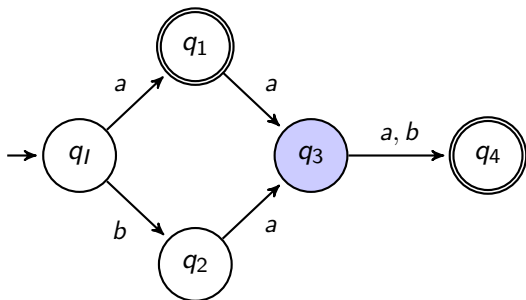
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There exists an algorithm that converts the minimum DFA accepting a reversible language into a REV-DFA

[Holzer&Jakobi&Kutrib '15]

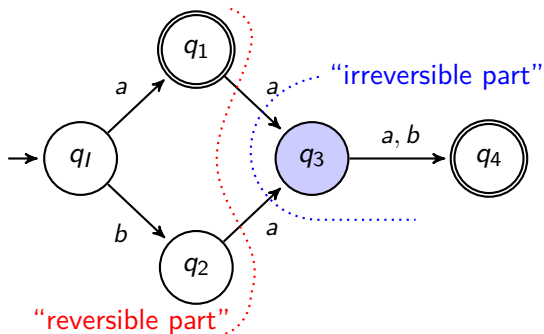
## From the minimum DFA to an equivalent REV-DFA

The minimum DFA  $M$  accepting  $L = \{a, aaa, aab, baa, bab\}$



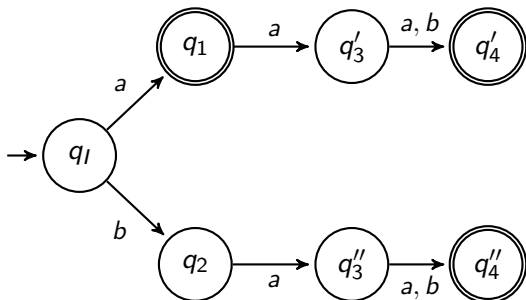
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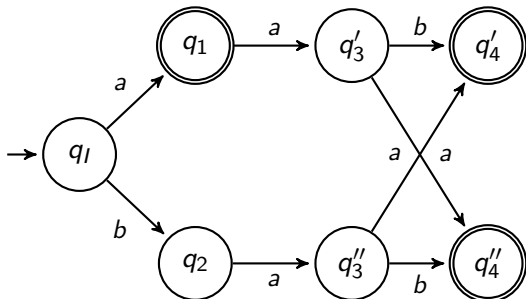
## From the minimum DFA to an equivalent REV-DFA

A minimal REV-DFA accepting  $L = \{a, aaa, aab, baa, bab\}$



## From the minimum DFA to an equivalent REV-DFA

In general a minimal REV-DFA is not unique



Another nonisomorphic minimal REV-DFA accepting  $L$



# Our Goal

Here we want to go beyond ...

We investigate the *existence of many minimal and reduced*  
REV-DFAs.

# Minimal Reversible Automata

# Minimal Reversible Automata

A characterization of languages having several different minimal REV-DFAS

We obtained the following characterization:

## Theorem

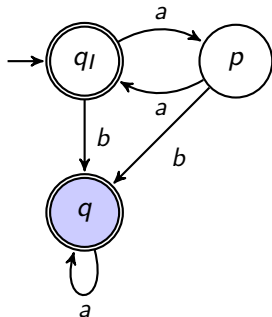
*Let  $M = (Q, \Sigma, \delta, q_I, F)$  be the minimum DFA accepting a reversible language  $L$ . The following statements are equivalent:*

- 1. There exists a state  $q \in Q$  in the irreversible part such that  $\delta^R(q, a) \neq \emptyset$ ,  $\delta^R(q, b) \neq \emptyset$ , for some  $a, b \in \Sigma$ , with  $a \neq b$ .*
- 2. There exist at least two minimal nonisomorphic REV-DFAS accepting  $L$ .*

# Minimal Reversible Automata

An example:  $L = (aa)^* + a^*ba^*$

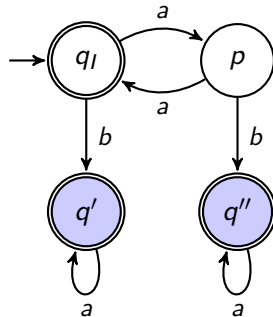
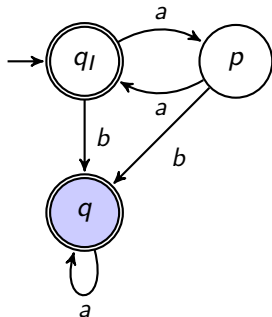
A minimum DFA  $M$  accepting a reversible language  $L$



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An equivalent minimal REV-DFA  $A'$ , obtained by the algorithm

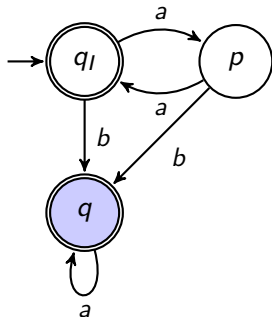


Minimum DFA  $M$

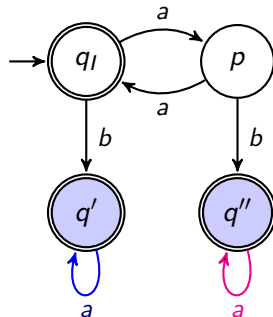
# Minimal Reversible Automata

An example:  $L = (aa)^* + a^*ba^*$

The idea is to use the set of states as in  $A'$  and to modify only the transitions which simulates the transitions that in  $M$  enter the state  $q$



Minimum DFA  $M$

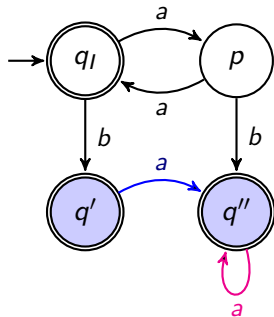
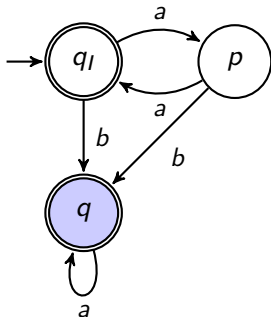


Minimal REV-DFA  $A'$

# Minimal Reversible Automata

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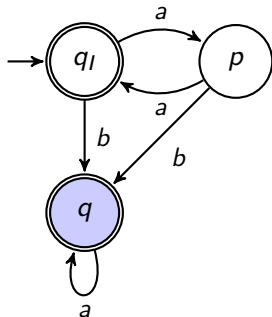


Minimum DFA  $M$

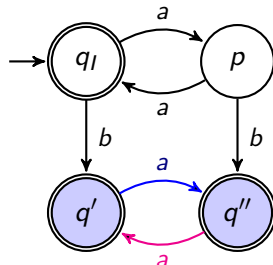
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Minimum DFA  $M$



Another minimal REV-DFA



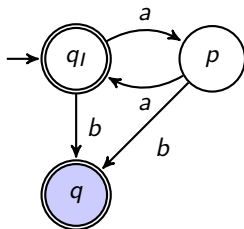
# Minimal Reversible Automata

A characterization of languages having a unique minimal REV-DFA

## Corollary

*There exists a unique (up to isomorphism) minimal REV-DFA accepting  $L$  if and only if for each  $q \in Q$  in the irreversible part of the minimum DFA, all the transitions entering in  $q$  are on the same symbol.*

The minimum DFA accepting  $L = (aa)^* + a^*b$ :



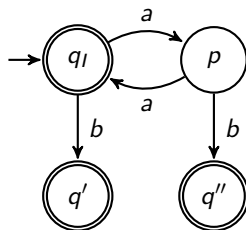
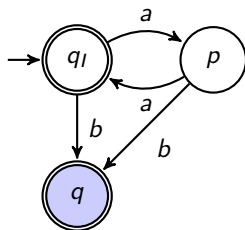
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The minimum DFA and the *unique* minimal REV-DFA:



# Minimal Reversible Automata

## Observations

Minimal  $\equiv$  Minimum number of states

$\Rightarrow$  Trying to merge two equivalent states in a minimal REV-DFA  $A$ , the resulting automaton is not reversible, i.e.,  $A$  is *reduced*

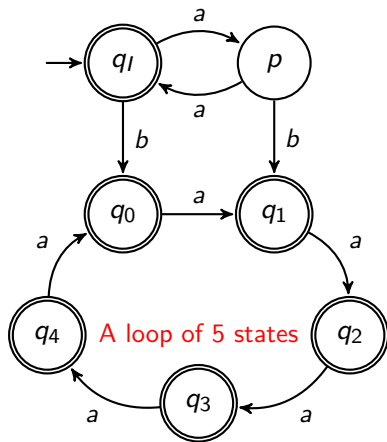
However,

there exist REV-DFAs which are reduced but not minimal!

# Reduced Reversible Automata

## Merging equivalent states: an example

$$L = (aa)^* + a^*ba^*$$

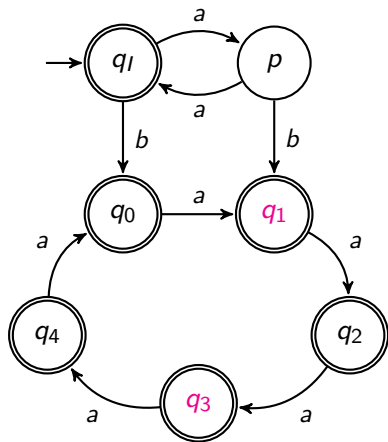


All states in the loop are equivalent.

Let us try to merge two of them, i.e.,  $q_1$  and  $q_3$

# Merging equivalent states: an example

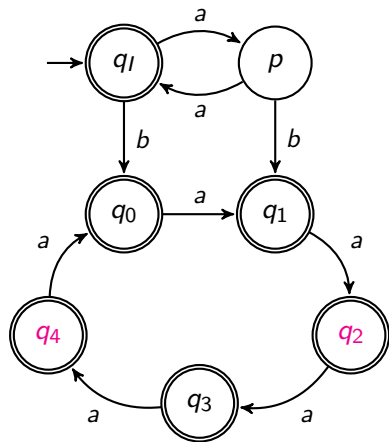
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$q_1 - q_3$

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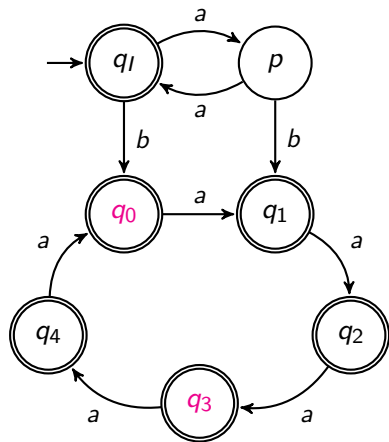


$q_1 - q_3$

$q_2 - q_4$

# Merging equivalent states: an example

$$L = (aa)^* + a^*ba^*$$



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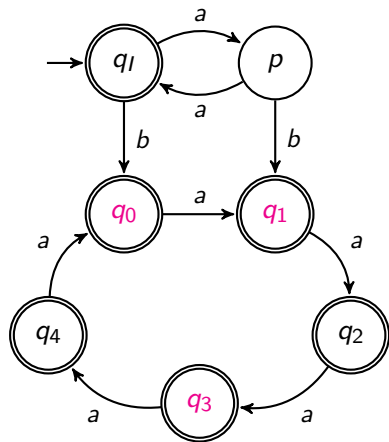
$q_2 - q_4$

$q_3 - q_0$



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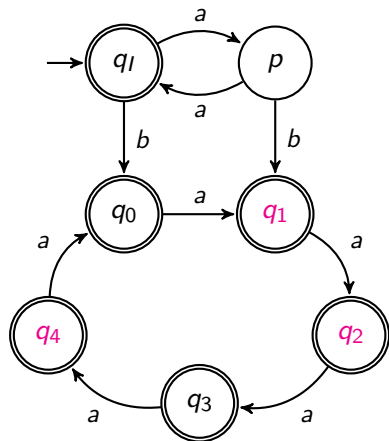
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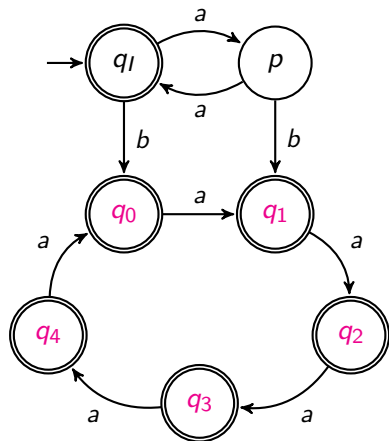
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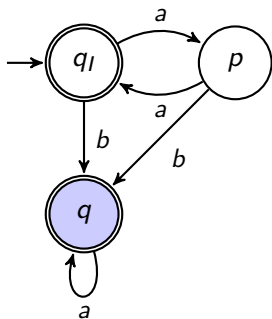
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$$q_3 - q_0 - q_1$$

$$q_4 - q_1 - q_2 - q_3 - q_0$$

## Merging equivalent states: an example

$$L = (aa)^* + a^*ba^*$$



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$$q_2 - q_4$$

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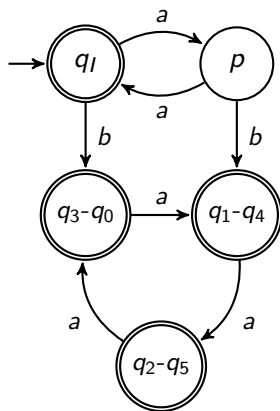
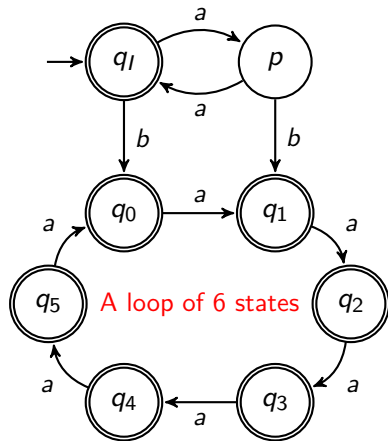
$$q_4 - q_1 - q_2 - q_3 - q_0 = q$$

The resulting automaton  
is not reversible!

A reversible automaton is said to be *reduced* if every automaton obtained from it by merging some equivalent states is not reversible.

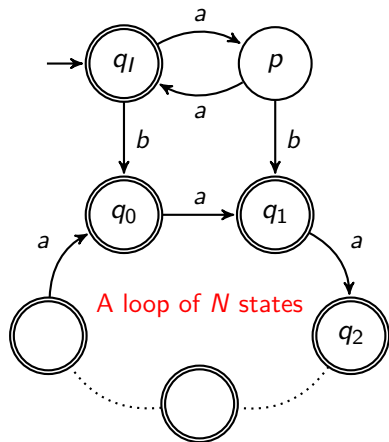
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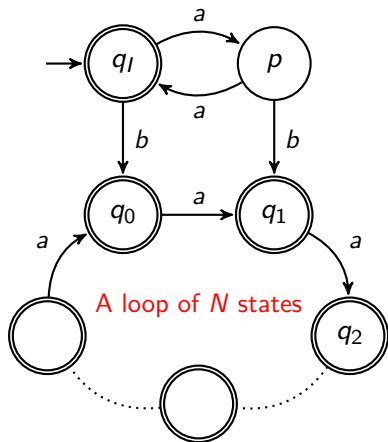
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The REV-DFA is reduced if and only if  $N$  is prime

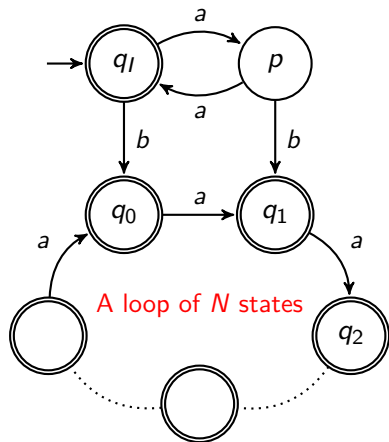
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- ▶ For each arbitrarily large  $N$ , there exists a reduced REV-DFA accepting  $L$  with more than  $N$  states

The REV-DFA is reduced if and only if  $N$  is prime

$$L = (aa)^* + a^*ba^*$$



- ▶ For each arbitrarily large  $N$ , there exists a reduced REV-DFA accepting  $L$  with more than  $N$  states
- ▶ There are infinitely many reduced equivalent REV-DFAS

The REV-DFA is reduced if and only if  $N$  is prime



# What about other reversible languages?

## Problem

Given a reversible language  $L$ , is it always possible to find arbitrarily large and hence infinitely many equivalent reduced reversible automata?

The answer is negative when  $L$  is finite.

What about  $L$  infinite?

# What about other reversible languages?

## Problem

Given a reversible language  $L$ , is it always possible to find arbitrarily large and hence infinitely many equivalent reduced reversible automata?

The answer is negative when  $L$  is finite.

What about  $L$  infinite?

# Reduced Reversible Automata

## A sufficient condition

- ▶  $L$  reversible language
- ▶  $M = (Q, \Sigma, \delta, q_I, F)$  minimum DFA accepting  $L$

### Theorem

*If  $M$  contains a loop in the irreversible part, then there exist infinitely many nonisomorphic reduced REV-DFAs accepting  $L$ .*

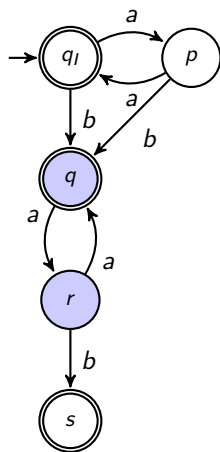
# Reduced Reversible Automata

## Proof idea

- ▶ A generalization of the argument in previous example
- ▶ Let  $N$  be a sufficient large prime number
- ▶ In a minimal REV-DFA modify the part corresponding to a nontrivial strongly connected component (SCC)  $\mathcal{C}$  in the irreversible part, with  $N$  copies of each state in  $\mathcal{C}$  and arranging the transitions in such a way that all the states in these  $N$  copies form a new SCC
- ▶ Furthermore, all the SCCs that follow  $\mathcal{C}$  will be replicated a certain number of times

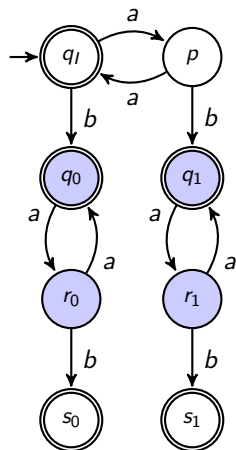
# Reduced Reversible Automata: other example

$$L = (aa)^* + a^*b + a^*b(aa)^*ab$$



A minimum DFA

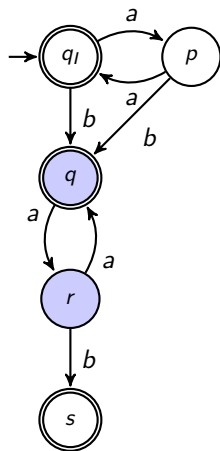
$N = 2$



An equivalent minimal REV-DFA

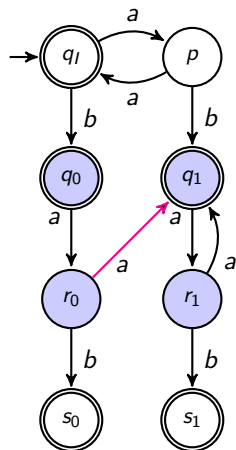
# Reduced Reversible Automata: other example

$$L = (aa)^* + a^*b + a^*b(aa)^*ab$$



A minimum DFA

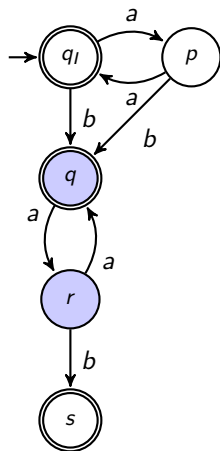
$N = 2$



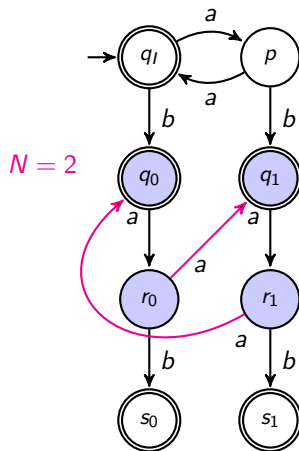
Modifying the transitions...

# Reduced Reversible Automata: other example

$$L = (aa)^* + a^*b + a^*b(aa)^*ab$$



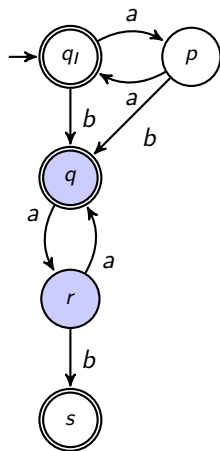
A minimum DFA



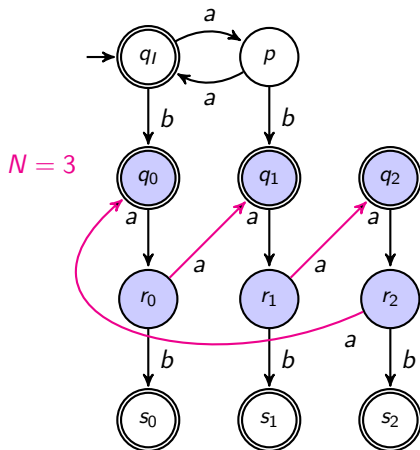
Another minimal REV-DFA for  $L$

# Reduced Reversible Automata: other example

$$L = (aa)^* + a^*b + a^*b(aa)^*ab$$



A minimum DFA



An equivalent reduced REV-DFA



# Reduced Reversible Automata

We proved a sufficient condition:

## Theorem

*If  $M$  contains a loop in the irreversible part, then there exist infinitely many nonisomorphic reduced REV-DFAs accepting  $L$ .*

The condition is not necessary

## Problem: left open!

Find a characterization of reversible languages having infinitely many reduced REV-DFAs

# Conclusion

# Our contributions

- ▶ A characterization of the languages having several different minimal reversible automata, in terms of the structure of the minimum DFA
- ▶ A sufficient condition for the existence of infinitely many reduced REV-DFAs accepting a same reversible language

# Future works

Some possible future investigations

It could be interesting to study other models:

- ▶ Unique initial state and a unique final state
- ▶ A set of initial states and a set of accepting states

Thank you for your attention!