Properties Going Up and Lying Over in Universal Algebras

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The study of properties Going Up and Lying Over originates in ring theory [8]; these properties play an important role in the study of field extensions and in class field theory. More recently, their study in algebraic structures related to non–classical logics, namely MV–algebras and lattice–orderred abelian groups, has been initiated [3]. We have introduced and studied these properties in the general setting of congruence–modular equational classes.

Let $\mathcal{C}$ be a congruence–modular equational class, so that the commutator, $[\cdot,\cdot]$, is defined in $\mathcal{C}$ [1, 6]. For any algebra A from $\mathcal{C}$, we denote by $\mathrm{Spec}(A)$ the set of the *prime congruences* of A, that is those congruences ϕ of A such that, for any congruences α,β of A, $[\alpha,\beta]\subseteq\phi$ implies $\alpha\subseteq\phi$ or $\beta\subseteq\phi$. Let A and B be members of $\mathcal{C}$, $f:A\rightarrow B$ be a morphism in $\mathcal{C}$, f^* be the inverse image of the morphism $f\times f:A\times A\rightarrow B\times B$ and $\mathrm{Ker}(f)$ be the kernel of f. We say that f is *admissible* iff $f^*(\mathrm{Spec}(B))\subseteq\mathrm{Spec}(A)$, where f^* denotes the direct image of the map f^*. In the case when f is admissible, we say that f fulfills property *Going Up* (abbreviated *GU*) iff, for all $\psi\in\mathrm{Spec}(B)$, $\{\phi\in\mathrm{Spec}(A)\mid f^*(\psi)\subseteq\phi\}\subseteq f^*(\{\chi\in\mathrm{Spec}(B)\mid\psi\subseteq\chi\})$, and we say that f fulfills property *Lying Over* (abbreviated *LO*) iff $\{\phi\in\mathrm{Spec}(A)\mid\mathrm{Ker}(f)\subseteq\phi\}\subseteq f^*(\mathrm{Spec}(B))$.

We list some of the results on these properties that we have obtained. GU implies LO. Surjectivity implies admissibility, but the converse is not true, for example there exist admissible bounded lattice morphisms which are not surjective. Admissibility doesn‘t always hold, for instance it doesn‘t hold for all bounded lattice morphisms. Admissibility does not imply LO, thus it does not imply GU, for instance not all admissible bounded lattice morphisms fulfill LO. The study of these properties can be reduced to the case of canonical embeddings: if i is the canonical embedding of $f(A)$ into B, then: f is admissible iff i is admissible, and, if these morphisms are admissible, then: f fulfills GU, respectively LO, iff i fulfills GU, respectively LO. Admissibility, GU and LO are preserved by composition, quotients, and finite direct products in the case when $\mathcal{C}$ has no skew congruences, in particular if $\mathcal{C}$ is congruence–distributive. If f is admissible, then: f fulfills GU iff the restriction $f\mid_{\mathrm{Spec}(B)}:\mathrm{Spec}(B)\rightarrow\mathrm{Spec}(A)$ is a closed map with respect to the Stone topologies. For what follows, note that, according to [9], each variety with equationally definable principal congruences (EDPC) is congruence–distributive, hence it has the commutator equal to the intersection of congruences [7]. We have proved that, in any variety with EDPC, all admissible morphisms fulfill GU, thus also LO; [4] provides many examples of varieties with EDPC. Varieties with EDPC include discriminator varieties, according to [7, Theorem 3.2, p. 389], and congruence–distributive varieties with the principal intersection property (PIP), by [5]. We have proved that, in discriminator varieties, as well as in congruence–distributive varieties with PIP, all morphisms are admissible, hence, by the above, all morphisms fulfill GU and LO. It has been proven that discriminator varieties include Boolean algebras, Post algebras, n–valued MV–algebras, monadic algebras, cylindric algebras, Gödel residuated lattices [10], while congruence–distributive varieties with PIP include distributive lattices [2, Example 4.7, p. 120]. It is well known that the variety of residuated lattices is congruence–distributive, and it is straightforward that it also has the PIP. Therefore all Boolean morphisms, all lattice morphisms between distributive lattices and all residuated lattice morphisms are admissible and fulfill GU, thus also LO, and the same holds for all morphisms in the varieties mentioned above.

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