Homomorphisms of abelian p-groups produce p-automatic recurrent sequences

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Abstract - If $\vec{v}_1, \dots, \vec{v}_m$ are lexicographically positive elements in $\mathbb{Z}^n$, a general recurrence is a computation rule $a(\vec{x}) = f(a(\vec{x}-\vec{v}_1), \dots, a(\vec{x}-\vec{v}_m))$. Let H be a finite abelian p-group, $h : H^m \to H$ a homomorphism of groups, $t \in H$ and $f : H^m \to H$ given by $f = h + t$. Suppose that one has an initial condition which is sufficient for the recurrence rule to uniquely define a n-dimensional sequence $a : \mathbb{N}^n \to H$ and this initial condition is given by p-automatic functions. Then the recurrent n-dimensional sequence $(a(\vec{x}))$ is p-automatic. This is a consequence of a result by Denef and Lipschitz connecting p-automatic sequences with formal series over p-adic integers $s \in \mathbb{Z}_p[[\vec{X}]]$ which are algebraic over the field of rational functions $\mathbb{Z}_p(\vec{X})$.

Key words and phrases : recurrent n-dimensional sequence, automatic sequences, context-free substitutions, $\mathbb{Z}_p[[\vec{X}]]$, $\mathbb{Z}_p(\vec{X})$, finite abelian p-groups, binomial coefficients modulo p^k, Lakhtakia-Passoja carpets modulo p^k, Sierpinski's Carpet.

Mathematics Subject Classification (2000) **:** 05B45, 28A80, 03D03.

Definition 0.1 *Fix $m \geq 1$ and m many tuples $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{Z}^n$ such that all m tuples are pairwise distinct and lexicographically positive: $\vec{v}_1 > \vec{0}$, ..., $\vec{v}_m > \vec{0}$. Such a collection of distinct lexicographically positive integral tuples is called a* system of predecessors *and will be denoted by $P = \{\vec{v}_1, \dots, \vec{v}_m\}$.*

Definition 0.2 *Let $m \geq 1$ and let $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{Z}^n$ such that $P = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a system of predecessors. Consider the set C_P defined as:*

$$C_P = \{\vec{x} \in \mathbb{N}^n \,|\, \exists\, i\ 1 \leq i \leq m \,\wedge\, \vec{x} - \vec{v}_i \in \mathbb{Z}^n \setminus \mathbb{N}^n\}.$$

Any function $c : C_P \to A$ is called initial condition *for the system of predecessors P. The set C_P is the* domain *of the initial condition.*

Definition 0.3 *Let A be a finite alphabet. Fix $m \geq 1$, a function $f : A^m \to A$ called rule of recurrence and $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{Z}^n$ a system of m distinct predecessors: $\vec{v}_1 > \vec{0}$, ..., $\vec{v}_m > \vec{0}$, denoted by $P = \{\vec{v}_1, \dots, \vec{v}_m\}$. Given an initial condition $c : C_P \to A$ for this system of predecessors, we say that an n-dimensional sequence $a : \mathbb{N}^n \to A$* satisfies the recurrence (A, f, P, c) *if and only if the following two conditions are fulfilled:*

1. *For all* $\vec{x} \in C_P$, $a(\vec{x}) = c(\vec{x})$.

2. *For all* $\vec{x} \in \mathbb{N}^n \setminus C_P$, $a(\vec{x}) = f(a(\vec{x} - \vec{v}_1), \ldots, a(\vec{x} - \vec{v}_m))$.

Definition 0.4 *Let* $d \geq 1$ *and* $s \geq 2$ *two natural numbers. A* n-dimensional system of substitutions *(for short,* n-dimensional substitution*) of type* $d \to sd$ *over the finite set* A *is a tuple of finite sets* $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$, *as follows:*

$\mathcal{D}$ *is a set of colored* n*-dimensional cubes* $D : \{0, \ldots, d-1\}^n \to A$,

$\mathcal{E}$ *is a set of colored* n*-dimensional cubes* $E : \{0, \ldots, 2d-1\}^n \to A$, *such that for every* $E \in \mathcal{E}$, $D_d(E) \subset \mathcal{D}$, *and*

$D_1 \in \mathcal{D}$ *is a special element called* start-symbol.

Finally, Σ *is a function* $\Sigma : \mathcal{D} \to \mathcal{E}$, *called the set of substitution* rules, *or simply the* substitution. *The function* Σ *has a natural extension defined on the set of cubes* F *such that* $D_d(F) \subseteq \mathcal{D}$. *We remark that if* $D_d(F) \subseteq \mathcal{D}$ *then* $D_d(\Sigma(F)) \subseteq \mathcal{D}$, *so* Σ *can be applied again to* $\Sigma(F)$. *Moreover,* Σ *must fulfill the following condition:*

$$\Sigma(D_1) \,|\, \{0, \ldots, d-1\}^n = D_1.$$

In this case, we say that the substitution Σ *is* expansive. *The number* $s \geq 2$ *is called the* factor of substitution.

Theorem 0.1 *Let* A *be a finite set, let* (A, f, P, c) *be an* n*-dimensional recurrence and let* $(A, \mathcal{D}, \mathcal{E}, D_1, \Sigma)$ *be an* n*-dimensional substitution. Suppose that the recurrence generates an* n*-dimensional sequence* $a : \mathbb{N}^n \to A$ *and that the substitution generates an* n*-dimensional sequence* $b : \mathbb{N}^n \to A$. *Finally, suppose that the substitution is of type* $d \to sd$ *and that the following conditions are satisfied:*

1. *If* R_P *is the minimal rectangle containing* $-P$ *and* $\{\vec{0}\}$, $k_1, \ldots, k_n$ *are the edge-lengths of* R_P *and* $k = \max(k_1, \ldots, k_n)$, *then* ($d = 1$ *and* $k = 2$) *or* $k \leq d$.

2. *For all* $\vec{x} \in C_P$, $a(\vec{x}) = b(\vec{x})$.

3. *There exists* $M \in \mathbb{N}$ *such that* $a \,|\, \{0, \ldots, ds^M - 1\}^n = \Sigma^M(D_1)$ *and* $C_d(\Sigma^{M-1}(D_1)) = C_d(\Sigma^M(D_1))$.

Then $a = b$.

Theorem 0.2 *Let* p *be a prime, and* $m \geq 1$. *Consider the group* $H = \mathbb{Z}/p^{d_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{d_s}\mathbb{Z}$ *and* $f : H^m \to H$ *a shifted homomorphism of groups. Let* $(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ *be an* n*-dimensional recurrence, such that the initial condition* $c : C_P \to H$ *is given by* p*-automatic sequences. This means that for all* $i = 1, \ldots, n$ *and for all* $a \in \mathbb{N}$, *if* $(x_i = a) \cap \mathbb{N}^n \subset C_P$, *then* $c \,|\, (x_i = a) \cap \mathbb{N}^n$ *is a* p*-automatic* $(n-1)$*-dimensional sequence. Then the recurrence* $(H, f, \vec{v}_1, \ldots, \vec{v}_m, c)$ *produces a* p*-automatic* n*-dimensional sequence.*

Corollary 0.1 *Every recurrent n-dimensional sequence over a finite abelian p-group, that is given by a shifted homomorphism, can be defined as limit sequence of a substitution of type $p^a \to p^b$ for some $a < b$. Moreover, there is an algorithm permitting to find the corresponding substitution.*

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