Reticulation of the hoops

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1 Abstract

In this paper we study the reticulation of a hoop. The reticulation is a bounded distributive lattice such that there exists a homeomorphism between the prime spectrum of the hoop and the prime spectrum of this lattice, endowed with the Stone topologies. We give an axiomatic definition of the reticulation and we prove its existence. We prove the uniqueness of the reticulation modulo a lattice isomorphism. We build a covariant functor between the category of bounded distributive lattices and the category of bounded hoops, which allows us to obtain new properties for hoops from the properties of lattices.

Definition 1.1 *A* hoop *is a structure* $(A, \leq, \cdot, \rightarrow, 1)$ *where* $(A, \leq)$ *is a poset with the greatest element* 1*,* $(A, \cdot, 1)$ *is a commutative monoid and :*

1. *For all* $x, y, z \in A$ *,* $x \leq y$ *implies* $x \cdot z \leq y \cdot z$ *and* $z \cdot x \leq z \cdot y$*.*
2. *For all* $x, y \in A$*,* $x \leq y$ *implies there is* $z \in A$ *such that* $x = y \cdot z$*.*
3. *For all* $x, y \in A$*,* $x \rightarrow y \in A$ *is the greatest element* $z \in A$ *such that* $z \cdot x \leq y$*.*

We only consider bounded hoops, which are hoops with a least element 0. For any $x, y \in A$, we define by: $x \wedge y = (x \rightarrow y) \cdot x = (y \rightarrow x) \cdot y$ and $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. With these operations $(A, \vee, \wedge)$ becomes a $\wedge$-semilattice but not necessary a lattice. A *filter* is a non-empty subset $F \subseteq A$ such that: for any $x, y \in A$ if $x, y \in F$ then $x \cdot y \in F$ and if $x \in F$ such that $x \leq y$ then $y \in F$. A filter P is *prime* iff for all $x, y \in A$, if $x \vee y \in P$ then $x \in P$ or $y \in P$. We define $D(X) = \{P \in Spec_F(A) | X \nsubseteq P\}$ for any $X \subseteq A$ and $D(x) = D(\{x\})$ for any $x \in A$, where $Spec_F(A)$ is the set of prime filters of A. $\{D(F) | F \in \mathcal{F}(A)\}$ is a topology on $Spec_F(A)$ and has the base $\{D(x) | x \in A\}$. The same goes in bounded distributive lattices.

Definition 1.2 *The reticulation of a hoop* A *is a pair* $(L(A), \lambda)$ *with* $L(A)$ *a bounded distributive lattice and* $\lambda : A \rightarrow L(A)$ *a surjective map, such that the function given by the inverse image of* λ *induces a homeomorphism between the prime spectrum of* $L(A)$ *and that of* A*.*

For a bounded distributive lattice L and a map $\lambda : A \to L$, we define the following conditions:

c1. For all $x, y \in A$, $\lambda(x \cdot y) = \lambda(x) \wedge \lambda(y)$;

c2. For all $x, y \in A$, $\lambda(x \vee y) = \lambda(x) \vee \lambda(y)$;

c3. $\lambda(0) = 0$; $\lambda(1) = 1$;

c4. λ is surjective;

c5. For all $x, y \in A$, $\lambda(x) \leq \lambda(y)$ iff $(\forall P \in Spec(A),\ x \in P \Rightarrow y \in P)$;

Proposition 1.1 *(L, λ) is a reticulation of A iff it verifies conditions c1-c5.*

For any $x, y \in A$ we denote: $x \equiv y$ iff $D(x) = D(y)$. Then $\equiv$ is a congruence relation on A with respect to the operations $\wedge$, $\cdot$ and $\vee$. For each $x \in A$, we denote by $[x]$ the congruence class of x with respect to $\equiv$. We consider on $A/\equiv$ the operations: $[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [x \vee y]$ for all $x, y \in A$. Then $(A/\equiv, \vee, \wedge, [0], [1])$ is a bounded distributive lattice. We consider the map $\lambda : A \to A/\equiv$ defined by $\lambda(x) = [x]$ for any $x \in A$. Since λ verifies conditions c1-c5, it follows that $(A/\equiv, \lambda)$ is a reticulation of A.

Proposition 1.2 *For any hoop A and any reticulations $(L_1, \lambda_1), (L_2, \lambda_2)$ of A, there exists an isomorphism of bounded distributive lattices $f : L_1 \to L_2$ such that $f \circ \lambda_1 = \lambda_2$.*

For any hoops A and B, a function $h : A \to B$ is called a *morphism of hoops* iff $h(0) = 0$, $h(1) = 1$ and for any $x, y \in A$, $h(x \cdot y) = h(x) \cdot h(y)$ and $h(x \to y) = h(x) \to h(y)$. For any hoop A, if $(L(A), \lambda_A)$ is the reticulation of A, then we denote by $\mathcal{L}(A) = L(A)$. For any hoops A, B, if $(L(A), \lambda_A)$, $(L(B), \lambda_B)$ are their reticulations and $h : A \to B$ is a morphism of hoops, then we denote by $\mathcal{L}(h)$ the map $\mathcal{L}(h) : L(A) \to L(B)$ defined by $\mathcal{L}(h)(\lambda_A(x)) = \lambda_B(h(x))$ for all $x \in A$. We denote by $\mathcal{H}0$ the category of bounded hoops and by $\mathcal{D}01$ the category of bounded distributive lattices.

Proposition 1.3 *$\mathcal{L} : \mathcal{H}0 \to \mathcal{D}01$ is a functor.*

We call $\mathcal{L}$ the *reticulation functor*. If we use the construction above for the reticulation of a hoop, then, for any hoops A, B and any hoop morphism $h : A \to B$, the reticulation functor shall be defined by: $\mathcal{L}(A) = A/\equiv$ and $\mathcal{L}(h)([x]) = [h(x)]$ for any $x \in A$.

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