Abstract
In mathematical programming of the n-set functions is considered a framework where the Kuhn-Tucker conditions are equality relations. For a multiobjective fractional program involving generalized $(\rho, b)$-vex n-set functions there is defined a multiobjective fractional program with equality constraints and weak, direct and converse duality theorems are established.

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1. INTRODUCTION

In 1979 Morris [9] developed the first optimization theory of the set functions. He defined the convexity and differentiability notions for the set functions and established optimal conditions and Lagrange duality for nonlinear programming problems with set functions. The Corley’s paper [2] is very important because he initiated the study of the functions of several set variables (n-set functions), gave the notions of partial derivative and of derivative of a n-set function and establishes some optimality conditions for mathematical programs involving n-set functions.

The research topic direction introduced by Corley was developed by Zalmai [12,13], Lin [4,5], Preda [10,11], I.M. Stancu-Minasian [11], Mititelu, Preda[7,8] etc.

Let $\Gamma^n$ be the n-fold product of a $\sigma$-algebra $\Gamma$ of subsets of a given set $X$ and the vector functions $f=(f_1, ..., f_p): \Gamma^n \rightarrow \mathbb{R}^p$, $g=(g_1, ..., g_p): \Gamma^n \rightarrow \mathbb{R}^p$, $h=(h_1, ..., h_q): \Gamma^n \rightarrow \mathbb{R}^q$ and $k=(k_1, ..., k_m): \Gamma^n \rightarrow \mathbb{R}^m$ ($'$ is the transposition sign). In this paper is presented a framework where the efficiency conditions of Kuhn-Tucker type
are equality relations. This idea is illustrated with the following multiobjective program generated by \( n \)-set functions:

\[
\begin{aligned}
\text{(PE)} & \begin{cases} 
\text{Minimize} & \left( f_1(S), \ldots, f_p(S) \right) \\
\text{subject to} & h(S) \leq 0, k(S) = 0, S \in \Gamma^n.
\end{cases}
\end{aligned}
\]

We consider the following multiobjective fractional program with mixed constraints:

\[
\begin{aligned}
\text{(PFE)} & \begin{cases} 
\text{Minimize} & \left( \frac{f_1(S)}{g_1(S)}, \ldots, \frac{f_p(S)}{g_p(S)} \right) \\
\text{subject to} & h(S) \leq 0, k(S) = 0, S \in \Gamma^n.
\end{cases}
\end{aligned}
\]

For the program (PFE) is developed a duality through weak, direct and converse duality theorems, where the main constraints of the dual program are equality relations. Generalized \((\rho, b)\)-vexity hypotheses for the functions of the program are used. These results are extended also for a multiobjective fractional program with inequality and equality (mixed) constraints involving \( n \)-set functions of the same type. We denote by \( D_{PFE} \) the domain of (PFE).

We remind that for the vectors \( u = (u_1, \ldots, u_n)' \) and \( v = (v_1, \ldots, v_n)' \) the relations \( u = v, \ u < v, \ u \leq v, \ u \leq v \) etc, are defined as

\[
\begin{aligned}
& u = v \iff u_i = v_i, \ i = 1, n; \\
& u < v \iff u_i < v_i, \ i = 1, n; \\
& u \leq v \iff u_i \leq v_i, \ i = 1, n; \\
& u \leq v \iff u_i \leq v_i \text{ and } u \neq v.
\end{aligned}
\]

We write \( u'v \) for the inner product \( \sum_{i=1}^n u_i v_i \) of \( u \) and \( v \), where \( ' \) is the transposition sign.

2. DEFINITIONS AND PRELIMINARIES

Let \((X, \Gamma, \mu)\) be a finite positive atomless measure space with \( L_1(X, \Gamma, \mu) \) separable. Consider the pseudometric space \((\Gamma^n, d)\), where \( \Gamma^n \) is the \( n \)-fold product of the \( \sigma \)-algebra \( \Gamma \) and \( d \) is the pseudometric on \( \Gamma^n \) defined by

\[
d(S, T) = \left( \sum_{k=1}^n (\mu(S_k \Delta T_k))^2 \right)^{1/2}.
\]

Here \( S = (S_1, \ldots, S_n), T = (T_1, \ldots, T_n) \) and \( \Delta \) denotes the symmetric difference.

The set \( \Omega \in \Gamma \) can be identified with its indicator function \( I_{\Omega} \in L_\infty(X, \Gamma, \mu) \subset L_1(X, \Gamma, \mu) \) and then the \( \sigma \)-algebra \( \Gamma \) is identified with the subset
\{I_\Omega \mid \Omega \in \Gamma \} \subset L_\infty(X, \Gamma, \mu). For \varphi \in L_1(X, \Gamma, \mu) and \Omega \in \Gamma the integral \int_{I_\Omega} \varphi d\mu will be denoted by \langle \varphi, I_\Omega \rangle.

**Definition 2.1** [9]. A set function \( \varphi : \Gamma \to \mathbb{R} \) is said to be **differentiable** at \( T \in \Omega \) if there exists \( D\varphi(T) \in L_1(X, \Gamma, \mu) \), called the derivative of \( \varphi \) at \( T \), such that

\[
\varphi(S) = \varphi(T) + \langle D\varphi(T), I_S - I_T \rangle + \psi(S, T)
\]

for all \( S \in \Gamma \), where \( \psi : \Gamma \times \Gamma \to \mathbb{R} \) is \( O(d(S, T)) \), that is,

\[\lim_{d(S, T) \to 0} \psi(S, T) / d(S, T) = 0.\]

**Definition 2.2** [2]. A \( n \)-set function \( F : \Gamma^n \to \mathbb{R} \) admits a **partial derivative** at \( S^0 = (S_1^0, ..., S_n^0) \) with respect to variable \( S_k (1 \leq k \leq n) \) if the function

\[
\varphi(S) = F(S_1^0, ..., S_{k-1}^0, S_k, S_{k+1}^0, ..., S_n^0)
\]

admits the derivative \( D\varphi(S_k^0) \), and we define \( D_k F(S^0) = D\varphi(S_k^0) \). The derivative of \( F \) at \( S^0 \) is \( DF(S^0) = (D_1 F(S^0), ..., D_n F(S^0)) \).

**Definition 2.3** [2]. For a vector set function \( f = (f_1, ..., f_p)' : \Gamma^n \to \mathbb{R}^p \), the **partial derivative** with respect to the variable \( S_k \) at \( S^0 \) is

\[
D_k f(S^0) = (D_1 f_1(S^0), ..., D_k f_k(S^0), ..., D_p f_p(S^0)).
\]

**Definition 2.4** [2] A \( n \)-set function \( F : \Gamma^n \to \mathbb{R} \) is differentiable at \( S^0 \) if there exists \( DF(S^0) \) and \( \psi : \Gamma^n \times \Gamma^n \to \mathbb{R} \) such that

\[
F(S) = F(S^0) + \sum_{k=1}^n \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle + \psi(S, S^0),
\]

where \( \psi(S, S^0) \) is \( O(d(S, S^0)) \).

**Definition 2.5** [11]. Let \( F : \Gamma^n \to \mathbb{R} \) differentiable at \( S^0, b : \Gamma^n \times \Gamma^n \to \mathbb{R}_+ \) and \( \rho \in \mathbb{R} \).

1) \( F \) is said to be \( (\rho, b) \)-**vex** [strictly \( (\rho, b) \)-vex] at \( S^0 \) if for all \( S \in \Gamma^n [S \neq S^0] \) we have

\[
b(S, S^0) \left| F(S) - F(S^0) \right| \geq [+] \sum_{k=1}^n \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle + \rho b(S, S^0) d^2(S, S^0)
\]

2) \( F \) is said to be **pseudo \( (\rho, b) \)-vex** [strictly pseudo \( (\rho, b) \)-vex] at \( S^0 \) if for all \( S \in \Gamma^n [S \neq S^0] \) we have:

\[
\sum_{k=1}^n \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle \geq \rho d^2(S, S^0) \Rightarrow b(S, S^0) F(S) \geq [+] b(S, S^0) F(S^0).
\]

3) \( F \) is **quasi \( (\rho, b) \)-vex** [strictly quasi \( (\rho, b) \)-vex] at \( S^0 \) if for all \( S \in \Gamma^n [S \neq S^0] \) we have:

\[
F(S) \leq F(S^0) \Rightarrow b(S, S^0) \sum_{k=1}^n \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle \leq [-] \rho b(S, S^0) d^2(S, S^0).
\]
4) [7] F is said to be monotonic quasi $(\rho, b)$-vex at $S^0$ if for all $S \in \Gamma^n$ we have

$$F(S) = F(S^0) \Rightarrow b(S, S^0) \sum_{k=1}^n \langle D_k F(S^0), I_{S_k} - I_{S^0_k} \rangle = -\rho b(S, S^0) d^2(S, S^0).$$

Let us denote by $D$ the domain of (PE) and let $P = \{1, \ldots, p\}, Q = \{1, \ldots, q\}$ and $M = \{1, \ldots, m\}$.

**Definition 2.6** [3] A point $S^0 \in D$ is an efficient solution (Pareto minimum) for (PE) if there exists no $S \in D, S \neq S^0,$ such that $f(S) \leq f(S^0)$.

For a multiobjective program with inequality constraints involving $n$-set functions Corley [2] defined the notion of the regular feasible solution and established necessary conditions for the efficiency of this solution. Mititelu and Preda adapted these two notions for the multiobjective program (PE) with mixed constraints as follows.

**Definition 2.7** [8] A point $S^0 \in D$ is a regular feasible solution for (PE) if $h$ and $k$ are differentiable at $S^0$ and there exists $T \in \Gamma^n$ such that for all $j \in Q$ and $s \in M$ we have

$$(R) \left\{ \begin{array}{l}
    h_j(S^0) + \sum_{k=1}^n \langle D_k h_j(S^0), I_{T_k} - I_{S^0_k} \rangle < 0, \\
    \sum_{s=1}^m \langle D_s k_s(S^0), I_{T_s} - I_{S^0_s} \rangle \leq 0.
\end{array} \right.$$ 

**Lemma 2.1** (Mititelu, Preda [8]). Let $S^0$ be a regular efficient solution of (PE) and let $f$, $h$ and $k$ be differentiable at $S^0$. Then there exist $u^0 \in \mathbb{R}^p, v^0 \in \mathbb{R}^q$ and $w^0 \in \mathbb{R}^m$ such that

$$\langle u^0, D_k f(S^0) + v^0, D_k h(S^0) + w^0, k(S^0), I_{S_k} - I_{S^0_k} \rangle \geq 0$$

\(\forall S_k \in \Gamma, 1 \leq k \leq n\)

$$u^0 \geq 0, u^0, e = 1, e = (1, \ldots, 1) \in \mathbb{R}^p$$

$$v^0 \geq 0, v^0, g(S) = 0, w^0, h(S) = 0.$$ 

In $n$-set programing these relations are considered efficiency conditions of Kuhn-Tucker type for the multiobjective program (PE) involving $n$-set functions.

**3. KUHN-TUCKER EFFICIENCY CONDITIONS AS EQUALITY RELATIONS**

In this section we establish efficiency conditions of Kuhn-Tucker type as equality relations for the program (PE) at a point $S^0 \in D$, using the regularity condition (R). The idea results from Theorem 2.2 by [6].

**Theorem 3.1** (Necessary conditions KT for (PE)[11]). Let $S^0 (\neq \emptyset, \neq X)$ be a regular efficient (or weakly efficient) solution of the type (R). We also suppose that the functions
\( f, h \) and \( k \) are differentiable at \( S^0 \). Then there exist vectors \( u^0 \in \mathbb{R}^p, v^0 \in \mathbb{R}^q \) and \( w^0 \in \mathbb{R}^m \) such that the following efficiency conditions of Kuhn-Tucker type for (PE) at \( S^0 \) are satisfied:

\[
\begin{align*}
&\left\{ \begin{array}{l}
u^0 \cdot D_k f(S^0) + v^0 \cdot D_k h(S^0) + w^0 \cdot D_k k(S^0) = 0, \quad k = 1, n \\
u^0 \cdot h(S^0) = 0, \quad v^0 \geq 0 \\
u^0 \geq 0, \quad e^T u^0 = 1.
\end{array} \right.
\end{align*}
\]

**Proof.** It is sufficient to analyze carefully the relation (2.1). We denote

\[
C_k = u_k \cdot D_k f(S^0) + v_k \cdot D_k h(S^0) + w_k \cdot D_k k(S^0)
\]

and then, the relation (2.1), for each \( k \), successively becomes

\[
\begin{align*}
&\langle C_k, I_{S_k} - I_{S_k^0} \rangle \geq 0, \quad \forall S_k \in \Gamma, \\
&\langle C_k, I_{S_k} \rangle \geq \langle C_k, I_{S_k^0} \rangle, \quad \forall S_k \in \Gamma, \\
&\int_{S_k} C_k d\mu \geq \int_{S_k^0} C_k d\mu, \quad \forall S_k \in \Gamma, \\
&C_k \mu(S_k) \geq C_k \mu(S_k^0), \quad \forall S_k \in \Gamma, \\
&C_k \left[ \mu(S_k) - \mu(S_k^0) \right] \geq 0, \quad \forall S_k \in \Gamma.
\end{align*}
\]

Particularly, for \( S_k = \emptyset \) we have \( \mu(S_k) = 0 \) and from relation (3.1) it results \( C_k \leq 0, \quad k = 1, n \). For \( S_k = X \) and having \( S_k^0 \subset X \), therefore \( \mu(S_k^0) < \mu(X) \), then from (3.1) it results \( C_k \geq 0, \quad k = 1, n \). Finally we obtain \( C_k = 0, \quad k = 1, n \) and so, the relation (2.1) is equivalent with the first relation of (KTE).

Zalmai established for the next multiobjective fractional program with inequality constraints:

\[
\begin{align*}
\text{(PF)} \quad &\text{Minimize} \left( \frac{f_1(S)}{g_1(S)}, ..., \frac{f_p(S)}{g_p(S)} \right) \\
\text{subject to} \quad &h(S) \leq 0, \quad S \in \Gamma^n
\end{align*}
\]

((PFE) without the constraint \( k(S) = 0 \)) the following necessary efficiency conditions:

**Lemma 3.1** (Zalmai[13, 2002]). Assume that \( f_i, g_i, i \in P \) and \( h_j \in Q \) are differentiable at \( S^0 \in \Gamma^n \) and for each \( i \in P \) there exist \( S^i \in \Gamma^n \) such that

\[
h_j(S^0) + \sum_{k=1}^{n} \langle D_k h_j(S^0), I_{S_k} - I_{S_k^0} \rangle < 0
\]

and for each \( l \in P \setminus \{i\} \),

\[
\sum_{k=1}^{n} \langle g_i(S^0) D_k f_j(S^0) - f_i(S^0) D_k g_j(S^0), I_{S_k} - I_{S_k^0} \rangle < 0.
\]
If \( S^0 \) is an efficient solution of \((PF)\), then there exist \( u^0 \in \mathbb{R}^p \) and \( v^0 \in \mathbb{R}^q \) such that
\[
\sum_{k=1}^{n} \left( \sum_{i=1}^{p} u^0_i g_i(S^0)D_k f_i(S^0) - f_i(S^0)D_k g_i(S^0) \right) + \sum_{j=1}^{q} v^0_j D h(S^0), I_{S^0} - I_{S^0}^\epsilon \geq 0, \quad (3.2)
\]
\( \forall S \in \Gamma^u \)
\( v^0_j h_j(S^0) = 0, \quad u^0 \geq 0, \quad e'u^0 = 1, \quad v^0 \geq 0. \)

**Proposition 3.2** If \( S^0 (\neq \varnothing, \neq X) \) is an efficient solution of \((PF)\) then
\[
c_k = \sum_{i=1}^{p} u^0_i g_i(S^0)D_k f_i(S^0) - f_i(S^0)D_k g_i(S^0) + \sum_{j=1}^{q} v^0_j D h_j(S^0) = 0.
\]

**Proof.** Relation (3.2) becomes
\[
\sum_{k=1}^{n} \langle c_k, I_{S_k} - I_{S_k}^\epsilon \rangle \geq 0, \quad \forall S_k \in \Gamma.
\]  
(3.3)

Particularly, for \( S_1 = S_1^0, ..., S_{k-1} = S_{k-1}^0; S_{k+1} = S_{k+1}^0, ..., S_n = S_n^0 \) relation (3.3) becomes
\[
\langle c_k, I_{S_k} - I_{S_k}^\epsilon \rangle \geq 0, \quad \forall S_k \in \Gamma,
\]
which implies \( c_k = 0 \). But the index \( k \) is arbitrarily choosen, Then \( c_k = 0, \forall k = 1, n \).

**Remark** Relation (3.2) and \( c_k = 0, k = 1, n \) are equivalent.

Consequently, according to Definition 2.7, Lemma 2.1, Proposition 3.2 and Remark, Definition 2.7 and Lemma 3.1, adapted for the program (PFE), have the following forms

**Definition 2.8** A point \( S^0 \in D_{PFE} \) is a regular feasible solution in Zalmai’s sense for
(PFE) if \( f_i, g_i, i \in P, h_j, j \in Q \) and \( k_s, s \in M \) are differentiable at \( S^0 \in \Gamma^u \) and for each \( i \in P \) there exist \( S^0_i \in \Gamma^u \) such that
\[
h_j(S^0) + \sum_{k=1}^{n} \langle D_k h_j(S^0), I_{S^0} - I_{S^0}^\epsilon \rangle < 0, \quad \forall j \in Q,
\]
\[
\sum_{k=1}^{n} \langle D_k k_l(S^0), I_{S^0} - I_{S^0}^\epsilon \rangle \leq 0, \quad \forall l \in M
\]
and for each \( l \in P \setminus \{i\}, \)
\[
\sum_{k=1}^{n} \langle g_i(S^0)D_k f_i(S^0) - f_i(S^0)D_k g_i(S^0), I_{S^0} - I_{S^0}^\epsilon \rangle < 0.
\]

**Definition 2.9** A point \( S^0 \in D_{PFE} \) is said to be a nonsingular solution for (PFE) if \( S^0 \neq \varnothing \) and \( \neq X. \)
Theorem 3.3 (Necessary efficiency conditions for (PFE)). Assume that $f_i, g_i, i \in P$, $h_j, j \in Q$ and $k_s, s \in M$ are differentiable at $S^0 \in D_{PFE}$. If $S^0$ is a nonsingular, regular efficient solution for (PFE) in Zalmai’s sense, then there exist $u^0 \in \mathbb{R}^p, v^0 \in \mathbb{R}^q$ and $w^0 \in \mathbb{R}^m$ such that

\[
\begin{align*}
\sum_{i=1}^p u_i^0 [g_i(S^0)D_kf_i(S^0) - f_i(S^0)D_kg_i(S^0)] + \\
\sum_{j=1}^q v_j^0 D_k h_j(S^0) + \sum_{s=1}^m w_s^0 D_s k_s(S^0) = 0
\end{align*}
\]

\(1 \leq k \leq n, v_j^0 h_j(S^0) = 0, j \in Q
\]
\(u^0 \geq 0, e'u^0 = 1, v^0 \geq 0.
\]

The relations (KTFE) represent the efficiency conditions of Kuhn-Tucker type for (PFE) at $S^0$.

DUALITY BETWEEN (PFE) AND (DFE)

Let $\{Q_1, \ldots, Q_r\}$ be a partition of $Q$, that is $Q_\alpha \subseteq Q, Q_\alpha \cap Q_\beta = \emptyset$ if $\alpha \neq \beta$, $\bigcup_{\alpha=1}^r Q_\alpha = Q$ and $\{M_1, \ldots, M_r\}$ a similar partition of $M$.

We suppose that the functions $f_i, g_i, i \in P, h_j, j \in Q$ and $k_s, s \in M$ are differentiable on $\Gamma^n$. Then we associate to (PFE) the following dual multiobjective fractional program of maximum Pareto

\[
\begin{align*}
\text{Maximize} & \quad \left( \frac{f_i(T)}{g_i(T)} , \ldots, \frac{f_p(T)}{g_p(T)} \right) \\
\text{subject to} & \quad \\
\sum_{i=1}^p u_i [g_i(T)D_k f_i(T) - f_i(T)D_k g_i(T)] + \\
\sum_{j=1}^q v_j^\alpha D_k h_j(T) + \sum_{s=1}^m w_s^\alpha D_s k_s(T) = 0
\end{align*}
\]

\(1 \leq k \leq n, v_j^\alpha h_j(T) = 0, j \in Q_\alpha
\]
\(T \in \Gamma^n, u \geq 0, \ e'u = 1, \ v \geq 0, \)

where

\[ v_j^\alpha h_j(T) = \sum_{j \in Q_\alpha} v_j h_j(T), \quad w_s^\alpha k_s(T) = \sum_{s \in M_\alpha} w_s k_s(T). \]

We denote by $\pi(S)$ the value of the primal program (PFE), by $\delta(T,u,v,w)$ the value of the dual program (DFE) and be $D_{DFE}$ the domain of (DFE). In what follows we
develop a duality relation between the pair of multiobjective fractional programs (PFE) and (DFE) with weak, direct and converse duality theorems.

**Theorem 4.1** (Weak duality). Let $S$ and $(T, u, v, w)$ be arbitrary feasible solutions of (PFE) and (DFE). Assume that:

a) For each $i \in P$, $f_i(T) > 0$, $g_i(T) > 0$, $g_i(S) > 0$;
b) For each $i \in P$, $f_i$ is pseudo $(\rho^*_i, b)$-vex at $T$, $-g_i$ is pseudo $(\rho^*_i, b)$-vex at $T$;
c) For each $\alpha = \overline{1, r}$, $v'_{Q_\alpha} h_{Q_\alpha}$ is quasi $(\rho^*_\alpha, b)$-vex at $T$;
d) For each $\alpha = \overline{1, r}$, $w^*_M k_M$ is monotonic quasi $(\rho^*_\alpha, b)$-vex at $T$;
e) One of the functions $f_i, -g_i$, $\forall i \in P$ is strictly pseudo $(\rho, b)$-vex $(\rho = \rho^*_i, \text{ or } = \rho^*_i)$, or one of $v'_{Q_\alpha} h_{Q_\alpha}$, $\forall \alpha$, is strictly quasi $(\rho^*, b)$-vex at $T$;
f) $\sum_{i=1}^{n} \left[ \rho^*_i g_i(T) + \rho^*_i f_i(T) \right] + \sum_{\alpha=1}^{r} (\rho^*_\alpha + \rho^*_\alpha^4) \geq 0$.

Then the relation $\pi(S) \leq \delta(T, u, v, w)$ is false.

**Proof.** From hypothesis b) it results:

$$\sum_{k=1}^{n} \langle D_k f_i(T), I_{S_i} - I_{T_i} \rangle \geq -\rho_i^4 d^2(S, T) \Rightarrow b(S, T) f_i(S) \geq b(S, T) f_i(T), \quad (4.1)$$

$$\sum_{k=1}^{n} \langle -D_k g_i(T), I_{S_i} - I_{T_i} \rangle \geq -\rho^*_i d^2(S < T) \Rightarrow -b(S, T) g_i(S) \geq -b(S, T) g_i(T), \quad (4.2)$$

For each $\alpha = \overline{1, r}$, according to c), we obtain

$$v'_{Q_\alpha} h_{Q_\alpha} (S) \leq v'_{Q_\alpha} h_{Q_\alpha} (T) \Rightarrow b(S, T) \sum_{k=1}^{n} \langle D_k v'_{Q_\alpha} h_{Q_\alpha}(T), I_{S_i} - I_{T_i} \rangle \leq -\rho^*_\alpha b(S, T) d^2(S, T) \quad (4.3)$$

For each $\alpha = \overline{1, r}$, according to d), it results

$$w^*_M k_k (S) = w^*_M k_k (T) \Rightarrow b(S, T) \sum_{k=1}^{n} \langle D_k w^*_M k_k (T), I_{S_i} - I_{T_i} \rangle = -\rho^*_\alpha b(S, T) d^2(S, T) \quad (4.4)$$

Taking into account the hypothesis e), for $S \neq T$, one of the second implications by (4.1), (4.2) or (4.3) is strictly; then means that $b(S, T) > 0$. In that follows, we consider these implications without the factor $b(S, T)$. Then, equivalently, we have

$$f_i(S) < f_i(T) \Rightarrow \sum_{j=1}^{p} \langle D_j f_i(T), I_{S_i} - I_{T_i} \rangle < -\rho_i^4 d^2(S, T), \quad (4.5)$$

$$-g_i(S) < -g_i(T) \Rightarrow \sum_{k=1}^{n} \langle -D_k g_i(T), I_{S_i} - I_{T_i} \rangle < -\rho_i^* d^2(S, T). \quad (4.6)$$
Now, we multiply (4.5) by \( u_i g_i(T) \geq 0 \) (but \( u'g(T) > 0 \)) and summ by \( i \in P \), multiply (4.6) by \( u_i f_i(T) \geq 0 \) (\( u'f(T) > 0 \)) and summ by \( i \in P \) and then we summ, side by side, the two obtained inequalities. It results
\[
\sum_{i=1}^{p} u_i [g_i(S) - g_i(S) f_i(T)] < 0 \quad \Rightarrow 
\]
(4.7)
\[
\Rightarrow \sum_{k=1}^{n} \left( \sum_{i=1}^{p} u_i [g_i(T) D_k f_i(T) - f_i(T) D_k g_i(T), I_{S_i} - I_{T_i}] < -\sum_{i=1}^{p} u_i [\rho_i^e g_i(T) + \rho_i^e f_i(T)] d^2(S, T) \right)
\]
We summ now (4.3) by \( \alpha = \overline{1, r} \) and summ also (4.5) by \( \alpha = \overline{1, r} \) and then, we summ side by side, the two obtained inequalities. One obtain
\[
\sum_{a=1}^{r} \left[ (v'_o h_Q + w'_m k_{M_s}(S)) - (v'_o h_Q + w'_m k_{M_s}(T)) \right] \leq 0 \quad \Rightarrow 
\]
(4.8)
\[
\Rightarrow \sum_{k=1}^{n} \left( \sum_{a=1}^{r} \left( v'_o h_Q + w'_m k_{M_s}(T), I_{S_a} - I_{T_a} \right) \right) \leq -\sum_{a=1}^{r} \left( \rho_a^e + \rho^e_a \right) d^2(S, T).
\]
Now we summ, side by side, implications (4.7) and (4.8) and obtain:
\[
\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - f_i(T) g_i(S)] + \sum_{j=1}^{q} v_j [h_j(S) - h_j(T)] + \sum_{s=1}^{m} w_k [k(S) - k(T)] < 0 \quad \Rightarrow 
\]
(4.8)
\[
\Rightarrow \sum_{k=1}^{n} \left( \sum_{i=1}^{p} u_i [g_i(T) D_k f_i(S) - f_i(T) D_k g_i(T)] + \sum_{j=1}^{q} v_j D_k h_j(T) + \sum_{s=1}^{m} w_k D_s k_s, I_{S_a} - I_{T_a} \right) <
\]
\[
< \left( \sum_{i=1}^{p} u_i [\hat{\rho}_i g_i(T) + \hat{\rho}_i f(T)] + \sum_{a=1}^{r} \left( \rho_a^e + \rho^e_a \right) \right) d^2(S, T).
\]
Taking now into account the first constraint of the dual (DFE) and the hypothesis f), the second inequality by (4.9) becomes \( 0 < 0 \), that is a false. Then, it results that the first inequality of (4.9) is false too and consequently, we have
\[
\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - f_i(T) g_i(S)] + \sum_{j=1}^{q} v_j [h_j(S) - h_j(T)] + \sum_{s=1}^{m} w_k [k(S) - k(T)] \geq 0, \quad \forall u \geq 0 
\]
(4.10)
But taking into account the relations \( S \in D_{PFE} \) and \( (T, u, v, w) \in D_{DFE} \) we obtain:
\[
\sum_{i=1}^{p} u_i [f_i(S) g_i(T) - f_i(T) g_i(S)] \geq 0, \quad \forall u \geq 0, e'u = 1
\]
and having \( f_i(T) g_i(S) > 0, \forall i \in P \), we have
\[
\sum_{i=1}^{p} u_i g_i(S) g_i(T) \left( \frac{f_i(S)}{g_i(S)} - \frac{f_i(T)}{g_i(T)} \right) \geq 0, \quad \forall u \geq 0, e'u = 1.
\]
(4.11)
Corollary 4.1 (Weak duality). Let $S$ and $(T, u, v, w)$ be arbitrary feasible solutions of (PFE) and (DFE), respectively. Assume satisfied the conditions a), b) of the Theorem 4.1 and the followings:

c’) For each $\alpha = 1, r$, $\Gamma_\alpha = v_0' Q_\alpha + w'_M s \alpha$ is quasi $(\rho_\alpha^*, b)$-vex at $T$;

d’) one of the functions $f_i, -g_i$, $\forall i \in P$ is strictly pseudo $(\rho, b)$ -vex $(\rho = \rho_i', \rho_s^*)$ at $T$,
or one of $\Gamma_\alpha, \forall \alpha$, is strictly quasi $(\rho_\alpha^*, b)$-vex at $T$;

f) $\sum_{i=1}^{p} u_i [\rho_i' f_i(T) + \rho_s^* f_j(T)] + \sum_{a=1}^{r} \rho_a^* \geq 0$.

Then the relation $\pi(S) \leq \delta(T, u, v, w)$ is not true.

Theorem 4.2 (Direct duality). Let $S^0$ be a nonsingular regular efficient solution of (PFE) in Zalmai’s sense and suppose satisfied the hypotheses of Theorem 4.1. Then there are vectors $u^0 \in \mathbb{R}^p$, $v^0 \in \mathbb{R}^q$ and $w^0 \in \mathbb{R}^m$ such that $(S^0, u^0, v^0, w^0)$ is an efficient solution for the dual (DFE) and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Proof. Because $S^0$ is a regular efficient solution of (PFE), according to Theorem 3.3 there are vectors $u^0 \in \mathbb{R}^p$, $v^0 \in \mathbb{R}^q$ and $w^0 \in \mathbb{R}^m$ such that the following relations are satisfied:

$$
\sum_{i=1}^{p} u_i \{g_i(S^0)D_k f(S^0) - f_i(S^0)D_k g_i(S^0)\} + \sum_{j=1}^{q} v_j^0 D_k h_j(S^0) + \sum_{i=1}^{m} w_s^0 D_k k_i = 0,
$$

$$
1 \leq k \leq n, \quad v_j^0 h_j(S^0) = 0, \quad \forall j \in Q,
$$

$$
u^0 \geq 0, \quad e^a u^0 = 1, \quad v^0 \geq 0.
$$

Also $w_i^0 k_i (S^0) = 0, \forall s \in M$. Then from these relations it results that $(S^0, u^0, v^0, w^0) \in D_{DFE}$ and in addition,

$$
\pi(S^0) = \left( \frac{f(S^0)}{g_i(S^0)}, ..., \frac{f_u(S^0)}{g_1(S^0)} \right) = \delta(S^0, u^0, v^0, w^0).
$$

Because Theorem 4.2 contains the hypotheses of Theorem 4.1, the relation $\pi(S^0) \leq \delta(S^0, u^0, v^0, w^0)$ is false. It follows that $(S^0, u^0, v^0, w^0)$ is an efficient solution for (DFE).
Corollary 4.2 (Direct duality). Let $S^0$ be a nonsingular regular efficient solution of (PFE) and suppose satisfied the hypotheses of Corollary 4.1. Then there are vectors $u^0 \in \mathbb{R}^p, v^0 \in \mathbb{R}^q$ and $w^0 \in \mathbb{R}^m$ such that $(S^0, u^0, v^0, w^0)$ is an efficient solution for the dual program (DPE) and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Theorem 4.3 (Converse duality). Let $(S^0, u^0, v^0, w^0)$ be an efficient solution of the dual program (DPE) and suppose that:

i) $\overline{S}$ is a nonsingular regular efficient solution of the primal program (PFE).

a) For each $i \in P$, \( f_i(S^0) > 0 \), \( g_i(S^0) > 0 \)

b) For each $i \in P$, $f_i$ is pseudo $(\rho^*_i, b)$-vex at $S^0$ and $-g_i$ is pseudo $(\rho^*_i, b)$-vex at $S^0$.

c) For each $\alpha = 1, r$, $\nu^*_Q, h_Q$ is quasi $(\rho^*_\alpha, b)$-vex at $S^0$;

d) For each $\alpha = 1, r$, $w^*_M, k_M$ is monotonic quasi $(\rho^*_\alpha, b)$-vex at $S^0$.

e) One of the functions $f_i, -g_i$, \( \forall i \in P \) is strictly pseudo $(\rho, b)$-vex $(\rho = \rho^*_i, \rho^*_j)$ at $S^0$ or one of $\nu^*_Q, h_Q, \forall \alpha$, is strictly quasi $(\rho^*_\alpha, b)$-vex at $S^0$;

f) \( \sum_{i=1}^{p} u^0_i [\rho^*_i g_i(S^0) + \rho^*_i f_i(S^0)] + \sum_{\alpha=1}^{q} (\rho^*_\alpha + \rho^*_\alpha) \geq 0. \)

Then $\overline{S} = S^0$ and $\pi(S^0) = \delta(S^0, u^0, v^0, w^0)$.

Proof. Suppose, by absurdum, that $\overline{S} \neq S^0$ and we shall find a contradiction. Because $\overline{S}$ is a nonsingular regular efficient solution of (PFE) then, according to Theorem 3.3, there are vectors $\overline{u} \in \mathbb{R}^p, \overline{v} \in \mathbb{R}^q$ and $\overline{w} \in \mathbb{R}^m$ such that the next conditions of KTFE type are satisfied:

\[
\sum_{i=1}^{p} \overline{u}_i [g_i(\overline{S}) - f_j(\overline{S})]D_k f_j(\overline{S}) + \sum_{j=1}^{q} \overline{v}_j D_k h_j(\overline{S}) + \sum_{s=1}^{m} \overline{w}_s D_s k_s(\overline{S}) = 0
\]

\[
1 \leq k \leq n, \quad \overline{u} \geq 0, \quad e^t \overline{u} = 1, \quad \overline{v} \geq 0, \quad \overline{v}_j h_j(\overline{S}) = 0, \quad j \in Q.
\]

Also we have $\overline{w}_s k_s(\overline{S}) = 0, \forall s \in M$. Then $(\overline{S}, \overline{u}, \overline{v}, \overline{w}) \in D_{DFE}$. The conditions (1.0) -- (7.0) are particularly hypotheses of Theorem 4.1. Moreover, $\overline{S} \in D_{PFE}$ and $(S^0, u^0, v^0, w^0) \in D_{DFE}$. Following the proof of Theorem 4.1 we obtain that the relation $\pi(\overline{S}) \leq \delta(S^0, u^0, v^0, w^0)$ is false. Moreover, $\pi(\overline{S}) = \delta(\overline{S}, \overline{u}, \overline{v}, \overline{w})$. Therefore the relation $\delta(\overline{S}, \overline{u}, \overline{v}, \overline{w}) \leq \delta(S^0, u^0, v^0, w^0)$ is false. Then the maximal efficiency of $(S^0, u^0, v^0, w^0)$ is contradicts. Therefore the supposition $\overline{S} \neq S^0$, above made, is false. It follows $\overline{S} = S^0$ and $\pi(S^0) = \delta(S^0, u^0, v^0)$.
Corollary 4.3 (Converse duality). Let \((S^0, u^0, v^0, w^0)\) be an efficient solution of the dual program (DFE) and suppose satisfied the next conditions:

i) \(\overline{S}\) is a nonsingular regular efficient solution of the primal program (PFE);

\[a^0, b^0\) and \(f^0\) of Theorem 4.3;

c’’) For each \(\alpha = \overline{1,r}\), \(\Gamma_a = v_{Q_a} h_{Q_a} + w_{M_a} k_{M_a}\) is quasi \((\rho_a, b)\)-vex at \(S^0\);

d’) one of the functions \(f_i, g_i, \forall i \in P\) are strictly pseudo \((\rho, b)\)-vex \((\rho = \rho'_i, \rho''_i)\) at \(S^0\) or \(\Gamma_a, \forall \alpha\) is strictly quasi \((\rho_a, b)\)-vex at \(S\);

Then \(\overline{S} = S^0\) and \(\pi(S^0) = \delta(S^0, u^0, v^0, w^0)\).

BIBLIOGRAPHY